MATHEMATICS MAGAZINE

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ROBERT E. HORTON, Editor

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STATISTICAL DECISION PROCEDURES IN INDUSTRY
I. CONTROL CHARTS BY VARIABLES

EDWARD P. COLEMAN, University of California, Los Angeles

1.1. Introduction. During the Spring of 1961, the Western Region Quality Control Conference Program Committee invited the above named individual to organize a seminar for industry on the fundamental principles and applications of statistical decision procedures in industry. In March of that year, Ben K. Gold, John M. Howell and O. B. Moan joined him in presenting this seminar at the Disneyland Hotel. These authors are again glad to respond to the invitation of editor Robert E. Horton to offer a revision and condensation of this material to the readers of Mathematics Magazine in five articles: (I) Control Charts by Variables, (II) Control Charts by Attributes, (III) Acceptance Sampling by Attributes, (IV) Acceptance Sampling by Variables, and (V) The Management of Quality and Reliability. The articles will appear with equations, figures, tables and references numbered so as to be referred to in any later article.

A bibliography will be given at the end of each of the five articles and will be referred to by the corresponding number in brackets []. Those desiring to study the subject of Quality and Reliability further are encouraged to study some of these well-written references, although this will not be necessary during the reading of the five articles to appear in this magazine. The theoretical and practical subject matter in these articles and in the references cited are pitched at a level easily comprehensible to teachers, students and others interested for the first time in Statistical Methods. We believe that many high school teachers and other readers can get from these articles a valid understanding of many of the practical applications of these decision procedures when used for controlling the quality of manufactured product in industry. Diligent attention given to these practical concepts at the high school level of instruction can give the student a real motivation for a more serious study of the underlying principles a little later in life.

There has been a rather rapid spread of the practice of using probability and statistical methods as a basis for making decisions about quality and reliability in industry during and since World War II. In the manufacturing industry these decisions have been made generally about the quality of mass-produced product. The characteristic feature of all these decisions is that they are made on the basis of randomly selected observations in samples and are thus made in the face of uncertainty [1.9]. Making decisions in this manner about manufactured product has given rise to the subject of statistical quality control [1.6]. It may be of interest to mention that Walter A. Shewhart [1.7, 1.8] conceived and presented in 1924 in a Bell Telephone Laboratories document the basic concepts of his now famous Shewhart control chart as the first statistical tool for making decisions about quality in manufactured product.

1.2. What is meant by the control of quality. It may be well to recognize at the outset that terms relating to the control of quality of manufactured product mean many things to many people in industry. The term "statistical quality control" meant, soon after Shewhart introduced the concept, the application of the Shewhart control chart to quality problems in machine shops and in other fabrication or assembly operations during manufacture. Some people have omitted the word "statistical" from this expression and others have used instead the expression "industrial quality control". In any case the terms "quality control" and "industrial quality control" generally convey a somewhat broader class of decisions relating to quality of product in industry than merely in the manufacturing cycle. Today the expression "total quality control" is being used by many to indicate that attention is being given to all operations and decisions in any industrial organization which affect the quality and reliability of manufactured product whether this be in the specifications, design, production, acceptance or use. The interested reader is referred to the monthly Journal of the American Society for Quality Control [1.1]. which treats a broad spectrum of problems related to the control of quality in industry.

While all these somewhat ambiguous expressions about quality tend to confuse our concepts of what is meant by the control of quality in manufactured product, it is clear from even a casual reading of Shewhart's books [1.7, 1.8] that he was developing a system of *experimental inference* of quality in manufactured product. This experimental inference system for making decisions in modern industry has been so expanded in the last few years that such topics as operations research, systems engineering, reliability engineering, linear programming, and other topics have been developed for a very broad class of decisions and analyses in industry. The reader should keep in mind that the five articles to appear in this magazine will be confined to a discussion of objective decision procedures used widely in modern industry to control the quality of the products of industry during the manufacturing cycle and at the final acceptance inspection transactions between the manufacturer and customer.

1.3. Objective verifiable evidence of quality. The central and unifying characteristic of the Shewhart system of experimental inference is its reliance upon appropriate steps of the scientific method:

1) The performance of experiments or tests on items of actual product or equipments, and
2) the drawing of objective conclusions from these experiments about the quality in the totality of all product or equipment in a population or under consideration.

There exist today many general and specific official documents called specifications, which are issued by the United States Government or by large prime-contractors, which state essentially that decisions about the quality or reliability of the product or equipment so covered shall be based on objective, verifiable data and not on such subjective considerations as the "reputation" of a particular contractor or the precision of a given screw machine unless "reputation" and "preciseness" are in each case clearly defined and expressed in terms that lend themselves to precise verification procedures.

There are two general methods of obtaining this objective verifiable evidence or data when we are concerned with production and inspection problems [1.4]:

1) The method of variables or measurements. Some examples of quality characteristics or quantities which vary from item to item are length, diameter, resistance, current, capacitance, tensile strength, time-to-failure, weight, cost and many others. There are many items of product, components and equipment systems which possess more than one of these characteristics, each of which would be treated individually.
2) The method of attributes or enumeration. Some examples of the method of attributes or enumeration are determining whether or not a unit or product functions as designed, a lamp lights or fails to light, the color or taste of something is satisfactory or not, a fuse blows or does not, a part passes or fails to pass a certain go-no-go gage, an employee produces the standard number of parts or he does not, an inspected component from a vendor is defective or non-defective and many other examples. There are many items of product, components and equipment systems which may possess more than one of these sources of success or failure in quality.

The essential feature or characteristic of objective verifiable data is that there must be an experimental or test basis for decisions made about quality of product. Experiments relating to quality of manufactured product are sometimes called statistical experiments. The distinguishing features of a statistical experiment are the random selection or drawing and the actual engineering measurement of the items drawn:

Random experiment. A random experiment deals with the selection of the sample items of product on which to base a decision about the quality of the manufacturing process or about the total population of product under consideration. In a random experiment, emphasis is placed on the method of selecting individual objects for measurement or test. The essential characteristic of the random selection process is that each item in the entire group or process must be given an equal chance or likelihood of being drawn in the sample. Each single item is given an equal chance of being drawn in the sample by placing the total collection of items in a container, say, stirring thoroughly and then having a blindfolded person select the first item. A second item can be selected randomly by returning the first item to the collection and repeating the random experiment until the desired number of items have been drawn. Repetitions of the random experiment will lead to the selection of different, and sometimes the same, items at each drawing. Random experiments are usually designed to be repeated under the same or controlled physical conditions. One can judge whether or not an item has been drawn randomly only by observing the process by which the item was drawn and not by its distinguishable characteristics observed after drawing. However, the three main features of total population (availability, stirring, blindfolded drawing in a random experiment) can be simulated by use of a random number table and by other means.

Engineering experiment. An engineering experiment deals with the measurement or identification of an item product which has been selected as a basis for the decision about the quality of a manufacturing process or of a given collection of items of product. In an engineering experiment, emphasis is placed upon the actual method of observing, measuring or identifying the item of product to be measured. The objective of an engineering experiment on an item of product is to obtain a quantitative value or values for each item of product selected. Consideration must be given to the units of measure and the degree of accuracy in the measuring equipment. It is clear that the measurement procedure of an engineering experiment may range from a simple visual inspection requiring only a fraction of a second of time to the operation of expensive and complicated measuring equipment requiring many hours to complete. Repetitions of an engineering experiment on a given item of product should lead to the same quantitative value being observed for each characteristic of quantity measured. Care, calibration and maintenance of the measuring equipment should be such as to assure this. For this reason, an engineering experiment on the same item of product is not usually repeated. When such repetitions are conducted, the object is to study variations in the measuring equipment.

1.4. Basic terms and concepts in quality decisions. Quality in manufactured products is always subject to certain amounts of variation which result from known and chance causes. According to Grant [1.6], some stable system of chance causes is inherent in any particular scheme of production and inspection. Variation within this stable pattern is inevitable. The cause for variation outside this stable pattern may be identified and reduced, corrected, eliminated or rejected. In rendering decisions about quality in a manufacturing process or its products, there are certain basic concepts that must be clearly defined and understood:

1) An item or unit of product
2) The quality characteristic, or quantity sometimes called the random variable, which varies from item to item of product
3) The population of all items of product about which a quality decision is to be made
4) The sample of items of product selected randomly from the population and on the basis of which the quality decision will be made
5) A population parameter whose value indicates the quality status of the population. Such parameters usually are those that indicate location of central value and dispersion or variation in the total

population of items being studied

6) The sample statistic, which is a function of the observations of the items in a sample, used to estimate the value of an unknown population parameter
7) A knowledge of what constitutes an acceptable state of quality in population and the allowable riske of wrong decision about quality in the population.

1.5. A statistical decision procedure for quality. A typical statistical decision procedure for quality in manufactured product usually involves seven distinct steps:

1) Definition of basic terms and concepts to be used in the decision procedure
2) Agreement on accepted facts about the population under study
3) State in statistical terms the quality decision problem
4) Fixed levels of risk, sample size, etc.
5) Choose the sample statistic on which to base decision
6) Find critical values for the quality decision procedures
7) Perform the statistical experiment and make the required decision about the process or population.

1.6. Shewhart control chart by variables. The Shewhart control chart is fundamentally a device for making decisions about the quality of product being turned out by a manufacturing process. The primary purpose of this control chart is to detect whether or not the variations in the product being produced lie within a stable system of chance causes and to detect any causes for variation outside of this stable chance cause system. It provides objective evidence in this manner for making one of two decisions concerning the quality of the production process at any specified time:

(1.1) H_0: The production process is in an acceptable state of statistical control.

(1.2) H_1: The production process is not in an acceptable state of statistical control.

Separate control charts are kept for each quality characteristic of the items of product in the population under study. Control charts by variables are introduced in this first article and some further examples will be given in the second article along with the introduction of control charts by attributes.

Control charts by variables are used in manufacturing operations to detect if the process has undergone a significant change in the mean value μ_x or in the standard deviation σ_x of the quality characteristic in the populations of units being produced. The procedure adopted in American industry is that at any time a random sample may be drawn from the process and the sample mean $\bar{x}$ and sample range R plotted on separate charts, where

$$\text{Random sample on } x:\ x_1,\ x_2,\ \cdots,\ x_i,\ \cdots,\ x_n\,. \tag{1.3}$$

$$(1.4)\qquad \bar{x} = \frac{x_1 + \cdots + x_n}{n}.$$

$$(1.5)\qquad R = x_{max} - x_{min}.$$

On each control chart, the abscissa, or horizontal coordinate, represents the time the sample was taken or the serial number of the sample, while the ordinate, or vertical coordinate, represents the values of $\bar{x}$ and R for that sample. If a stable system of chance causes is operating so that (1.1) is true, the points on each chart will form a random pattern about a center line, called the grand average, and two other horizontal lines, called the upper and lower control limits (UCL and LCL) for each statistic. These control limits are sometimes referred to as the 3-sigma limits and are drawn such that approximately 99.7% of all possible values of each statistic from the production process would lie between these lines. These control limits form the basis of making decisions about the state of the process at any time a sample is taken.

At a given time, if the sample points $\bar{x}$ and R on both charts are within the upper and lower control limits, the decision is to accept H_0 in (1.1); if otherwise, the decision is to accept H_1 in (1.2). The reader will note that at every time a decision is made, one of the following process status-decision situations exists :

		DECISION MADE	
		H_0	H_1
ACTUAL STATE OF PROCESS	H_0	Correct	Error
	H_1	Error	Correct

The proper use of a statistical decision procedure will enable one to control in the long run the risks of making wrong decisions. The following steps outline the procedure for setting up and using the $\bar{x}$ and R control chart system for making decisions about the state of statistical control and the quality of product from a manufacturing process when no prior knowledge of the process is known:

1) Let $x_1, \cdots, x_n$ be observations on x in random sample from process, where n is usually chosen to be between 3 and 10.
2) The sample mean and range are given by

$$(1.6)\qquad \bar{x} = \frac{x_1 + \cdots + x_n}{n}; \quad R = x_{max} - x_{min}.$$

3) The process grand average of k such sample means and ranges are given by

$$(1.7)\qquad \bar{\bar{x}} = \frac{\bar{x}_1 + \cdots + \bar{x}_k}{k}; \quad \bar{R} = \frac{R_1 + \cdots + R_k}{k},$$

where the number k is usually chosen to be between 20 and 30 and represents the first k samples of size n from the process after it has "settled down".

4) The lower and upper 3-sigma control limits for $\bar{x}$ are:

$$\text{LCL}_{\bar{x}} = \bar{\bar{x}} - A_2\bar{R} \;;\quad \text{UCL}_{\bar{x}} = \bar{\bar{x}} + A_2\bar{R} \;. \tag{1.8}$$

The constant A_2 for given sample size n is found in Table 1.1. The constants d_2, D_3 and D_4 below are also shown in this Table.

5) The lower and upper 3-sigma control limits for R are:

$$\text{LCL}_R = D_3\bar{R} \;;\quad \text{UCL}_R = D_4 R \;. \tag{1.9}$$

6) The lower and upper 3-sigma process tolerance limits for individual observations are

$$\text{LTL}_x = \bar{\bar{x}} - \frac{3\bar{R}}{d_2} \;;\quad \text{UTL}_x = \bar{\bar{x}} + \frac{3\bar{R}}{d_2} \;. \tag{1.10}$$

These last tolerance limits are not control limits, but they are the so-called "natural tolerance limits" of the process. If the process is in a state of statistical control, these values indicate the range of x-values between which approximately 99.7% of the items being produced will fall. Therefore, these natural tolerance limits for individual items of the process can be compared with the individual specification limits on x, which are usually found on the blueprint or drawing of the part. In making this comparison, if the natural tolerance limits of the process are within the specification limits for x, we conclude that almost all of the items of product are of satisfactory quality: if otherwise, the quality of the process is not satisfactory.

TABLE 1.1.

Factors for $\bar{x}$ and R Control Charts

Sample size n	d_2	A_2	D_3	D_4
2	1.128	1.880	0	3.267
3	1.693	1.023	0	2.575
4	2.059	0.729	0	2.282
5	2.326	0.577	0	2.115
6	2.534	0.483	0	3.004
7	2.704	0.419	0.076	1.924
8	2.847	0.373	0.138	1.864
9	2.970	0.337	0.184	1.816
10	3.078	0.308	0.223	1.777

1.7. Some statistical theory underlying control charts by variables. We have neither the space nor the intention in this series of articles to delve deeply into statistical theory, desirable as this might be from some standpoints. However, in order to understand what is actually happening during a control chart operation, it is necessary to mention a few important facts or assumptions from the theory of mathematical statistics:

1) The distribution of measurements x made on items of product from a process operating under the influence of a stable system of chance causes can be represented by the normal or Gaussian mathematical probability distribution function:

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{\left(-\frac{1}{2\sigma_x^2}(x-\mu_x)^2\right)}. \tag{1.11}$$

2) The normal probability distribution function (1.11) is dependent on the population mean μ_x and the population standard deviation σ_x as parameters. The graph of this function and the parameters are shown in Fig. 1.1. It is of interest to notice that the μ_x is the abscissa of the maximum point of the curve and that the points of inflection occur at a distance of σ_x from the ordinate at μ_x.

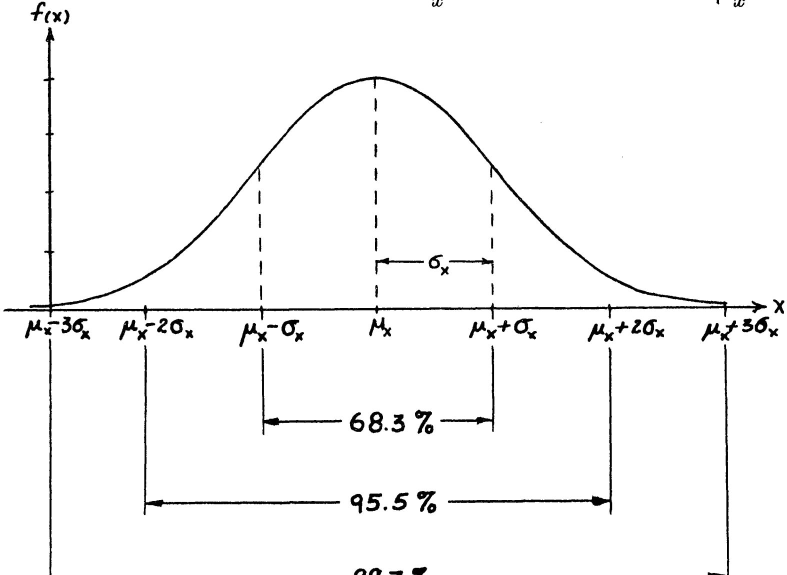

Fig. 1.1. The Normal Distribution

3) By a simple analysis of the function (1.11) and its graph in Fig. 1.1 one can observe that μ_x is the central value and σ_x a measure of dispersion in the population.

4) Being a probability distribution function for the population of individual values of x, the curve has the mathematical property that the probability of a randomly chosen value of x lying between any two fixed points a and b is given by the area under the curve between the ordinates a and b or $\int_a^b f(x)dx$.

5) By use of the property in 4) one can establish that approximately 68% of the area of the curve lies between the values $x = \mu_x \pm \sigma_x$; 95% between $x = \mu_x \pm 2\sigma_x$; and 99.7% between $x = \mu_x \pm 3\sigma_x$, these limits being the so-called 1-, 2-, and 3-sigma limits on x. This information is shown in Fig. 1.1.
6) If samples of size n are drawn repeatedly from the normal distribution, the probability distribution function of the sample means $\bar{x}$ is normal with population mean $\mu_{\bar{x}} = \mu_x$ and population standard deviation $\sigma_{\bar{x}} = \sigma_x/\sqrt{n}$. The probability distribution of $\bar{x}$ for various sample sizes is shown in Fig. 1.2.

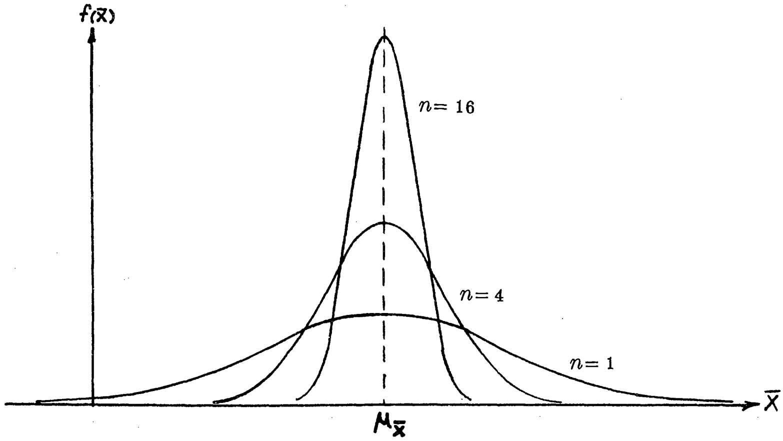

Fig. 1.2. Distribution of Means of Samples of Size n from the Normal Distribution

7) It is important to notice on the $\bar{x}$ control chart for any sample size n, that the center line is placed at the point $\bar{\bar{x}}$ which is an estimate of $\mu_{\bar{x}} = \mu_x$ and that the lower and upper control limits are drawn from the center line at a distance $A_2\bar{R}$ which is an estimate of $3\sigma_{\bar{x}} = 3\sigma_x/\sqrt{n}$.
8) The narrowing of the probability distribution curves for $\bar{x}$ in Fig. 1.2 as the sample size n increases is one of the reasons for using the control chart for $\bar{x}$ instead of for x. The statistic $\bar{x}$ is more sensitive to shifts in the population mean μ_x of the process than is x.
9) For a given production process in control the 3-sigma natural tolerance limits of the process are shown in Fig. 1.3 to be just within the specification limits. Fig. 1.3a shows that for a given sample size $n = 4$ essentially all or 99.7% of the $\bar{x}$'s fall between the control limits shown. However, Fig. 1.3b shows that a shift of $0.8\sigma_x$

in the population mean μ_x results in placing only 1% of the x's outside the original natural tolerance limits while 8% of the $\bar{x}$'s would lie outside the original control limits.

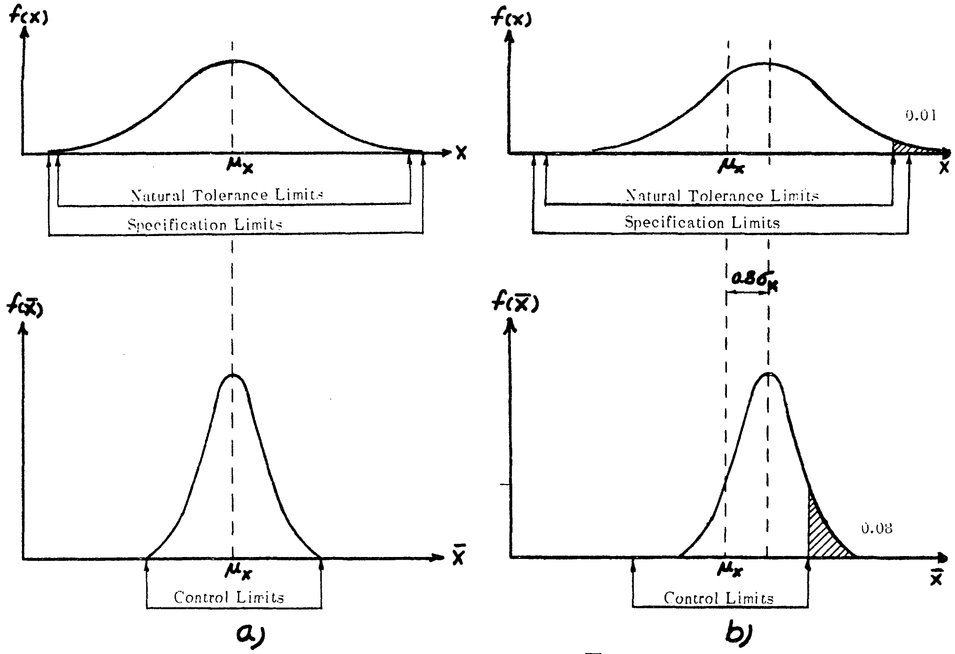

Fig. 1.3. Sensitivity of $\bar{x}$ Chart

10) Another interpretation of 9) is that the shaded areas to the right of $UCL_{\bar{x}}$ and UTL_x represent respectively the probability that a shift upward in μ_x will be detected by the $\bar{x}$ chart and the x chart. The sketch shows that for a shift upward in μ_x of $0.8\sigma_x$, the probability of this shift being detected by the x chart is only 0.01, while the probability of the shift being detected by the $\bar{x}$ chart is 0.08.
11) Intuitively one would think that a control chart for the sample standard deviation would be the appropriate chart for detecting a shift in the population standard deviation rather than a control chart on sample ranges. This in fact was done on the original Shewhart control charts. However, much industrial experience has revealed that the sample range R is almost as statistically efficient as the sample standard deviation and that the ease of calculating R in the shops is a highly desirable feature. This has almost rendered the control chart for the sample standard deviation as obsolete in practical industrial applications.
12) If it should be desirable or necessary to start a control chart before 20 to 30 samples have been drawn from a process and if

sufficient prior information is available, standard values for the parameters μ_x and σ_x of the underlying distribution can be assumed. For further discussion and examples of control charts with standard values, one should examine [1.2] and [1.3]. The essential characteristic of these charts is that the lower and upper control limits are placed at

$$\text{LCL}_{\bar{x}} = \mu_x - \frac{3\sigma_x}{\sqrt{n}}\,; \quad \text{UCL}_{\bar{x}} = \mu_x + \frac{3\sigma_x}{\sqrt{n}}\,. \tag{1.12}$$

13) The last remarkable and important property from statistical theory that needs to be mentioned here is as follows: Regardless of the parent population from which samples are drawn, as long as it has a mean μ_x and finite standard deviation σ_x, the probability distribution function of $\bar{x}$ from samples of size n approaches the normal probability function with mean $\mu_{\bar{x}} = \mu_x$ and standard deviation $\sigma_{\bar{x}} = \sigma_x/\sqrt{n}$, when n becomes large. In practice, the limiting distribution is good as an approximation even for sample sizes of five to ten. For a thorough treatment of the control-chart technique, including discussion of related statistical theory, the reader is referred to [1.5].

1.8. An example. The numerical observations presented in Table 1.2 represent measurements in order of production on the lengths of certain machined parts. These lengths are given in horizontal rows of 10 each and are the number of thousandths of an inch above and below 2.000 inches. The specification limits for this product are 2.000 ± 0.010 inches or for the numbers in Table 1.2, the specification limits are 0.00 ± 20 thousandths. For the purpose of this illustration, we shall assume that the observations made for the ten parts in each horizontal row were made in the half hours numbered serially from 1 to 100. We shall use the $\bar{x}$ and R control charts to control the quality of production in this machining operation. With the aid of the random numbers in Table 1.3, a sample of size $n = 3$ is drawn from each half hour of production.

The random sampling was done by associating the ten parts of each half hour's production with the ten digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 and then selecting three digits from the random number table 1.3 to determine the actual observations to include in the sample of size $n = 3$. The first hour's production was erratic and was not recorded during this so called "settling down" period for this machine, but samples of 3 were taken each half hour after this time. From the first 25 samples drawn from the machining process, the control limits were calculated as shown in Table 1.4 and placed on the $\bar{x}$ and R charts in Fig. 1.4. An inspection of these control charts revealed that all 25 sample points lie within the control limits for both $\bar{x}$ and R. Thus, the process was considered to be in a state of statistical control. The natural process tolerance limits were also calculated from the same 25 samples as shown in Table 1.4 and compared to

TABLE 1.2.

Lengths of 1000 Machined Parts

(Entire process is given in order of production, grouped by half hour intervals, ten pieces each half hour. Figures are number of thousandths of an inch above or below 2.000″.)

1	0	-5	-3	0	4	0	-1	-3	0	-7	51	8	-2	-14	-2	2	0	-4	4	16	-18
2	-5	-1	-4	1	0	4	0	-2	-2	11	52	10	-20	2	-4	-20	-30	-2	-2		-6
3	2	-1	8	-15	-9	5	1	-11	-10	-10	53	-10	-2	-10	-12	4	8	4	-4	4	-6
4	-3	1	-7	-4	-2	-10	-9	-10	-5	-2	54	4	-10	6	18	-4	0	-4	-10	4	16
5	-15	0	-1	0	-1	-4	0	-5	-3	-1	55	-12	-6	0	-28	0	1	4	0	-6	10
6	0	-1	-4	-6	-16	-10	7	-10	-9	9	56	-8	-18	-4	26	-2	-12	2	-8	6	-32
7	5	5	-2	-9	-9	-9	-5	-5	-3	-11	57	-24	-10	18	-4	2	-18	-2	-14	-8	-2
8	-14	-3	2	1	1	-2	3	-9	-11	-1	58	-24	-4	-34	12	-2	-36	-6	-18	-4	4
9	-1	-1	-4	-4	-11	-6	6	-10	-9	-1	59	-6	-8	-10	-12	-10	-16	-8	4	-12	10
10	2	-7	-2	-8	2	3	-2	-2	-8	-6	60	-4	-4	-8	4	0	-8	-10	-6	-4	-6
11	-5	-2	-5	-8	-1	7	-5	-1	-6	-8	61	20	6	30	2	0	-4	12	14	12	16
12	2	4	1	2	-2	-5	2	3	-3	2	62	6	-2	-2	-18	12	16	-12	-8	14	0
13	-5	0	3	-1	9	10	-2	2	-1	0	63	0	-8	12	10	-10	-12	8	-4	8	-10
14	2	-2	-5	-2	3	8	-4	-3	5	-3	64	12	10	-10	8	10	-4	2	-4	12	24
15	-6	-3	2	0	-14	1	0	-2	-3	0	65	-4	8	14	4	14	8	-18	8	-4	2
16	-2	2	5	-2	2	4	8	-6	-3	-5	66	18	-8	0	36	0	2	2	4	32	-12
17	-6	4	1	-1	-8	13	2	-4	-3	-2	67	14	6	10	24	20	2	-20	12	8	10
18	-3	-8	-3	-5	-2	-4	-6	-5	-7	-7	68	-4	-2	4	10	12	16	6	-2	-2	-18
19	-7	-7	3	-1	-2	-18	-18	1	-3	1	69	12	16	4	8	14	-4	14	8	-18	8
20	-6	2	-4	-5	-4	1	0	4	5	-3	70	6	-12	14	6	-8	22	14	0	0	-8
21	-3	1	-3	2	-1	-1	0	-1	3	2	71	0	4	1	5	2	-2	0	-2	-3	3
22	-6	-4	-7	-5	-6	-5	-1	-4	0	0	72	-6	9	-1	-15	-1	10	5	-1	4	2
23	-2	-5	-12	-7	-13	-5	-6	-5	-1	-4	73	1	-4	10	0	-9	4	3	-5	0	-2
24	2	7	-4	-5	-9	-1	-4	0	-3	4	74	-2	0	-2	-1	3	-4	0	5	4	5
25	3	1	-2	-5	6	3	-5	-6	3	-6	75	-3	0	3	1	3	3	-2	-1	-1	-5
26	-1	1	-2	-8	-3	2	-2	-4	-1	4	76	3	-7	-2	7	2	-5	-4	6	3	-6
27	2	-3	-3	1	-3	-4	-4	-6	-2	7	77	1	2	-4	0	12	-4	8	4	7	-5
28	-4	-16	-13	-6	-6	-7	-14	-9	-10	-7	78	1	10	4	-3	4	2	3	-2	-3	-4
29	-1	-3	-6	-1	6	3	7	0	4	6	79	0	3	-7	9	11	1	-1	0	-11	4
30	1	2	-3	-3	-6	2	-4	0	2	5	80	0	1	3	-1	-2	0	-5	4	-3	8
31	4	-8	-7	-6	-1	-7	-3	-6	-6	-3	81	3	-2	5	-1	2	-3	1	-4	11	0
32	9	4	-2	-5	-7	-4	-2	-2	1	5	82	6	-5	-9	-9	3	8	4	13	-1	-1
33	3	-4	0	-3	-3	1	1	1	1	1	83	1	-1	5	2	1	-2	9	5	-2	1
34	-14	-12	3	-17	2	2	2	0	5	6	84	-7	-7	0	4	3	3	3	3	7	-2
35	-4	-8	-2	-5	-4	-5	-4	-4	3	5	85	-1	6	4	4	6	5	-2	-1	-5	-1
36	0	3	-2	-6	-4	-8	-3	-5	-2	-1	86	4	-6	5	3	-2	4	-2	-4	1	-4
37	-5	-8	0	-6	-2	-3	-5	-7	0	-4	87	0	-3	-5	1	-6	3	7	-6	-2	10
38	1	-6	-2	0	-1	2	3	0	0	7	88	-2	-1	-1	0	-2	1	-1	2	5	5
39	-1	1	-2	-6	-5	-1	-4	4	-5	-7	89	-2	0	1	-2	0	-1	-1	-3	1	-4
40	1	-3	-1	-4	-1	-4	-8	-8	-1	0	90	3	4	1	3	0	0	0	6	1	14
41	7	0	3	8	2	-3	6	9	16	8	91	-8	-6	5	-2	-2	-6	4	3	3	4
42	7	2	4	1	5	2	0	7	3	0	92	-7	2	-4	2	-8	-7	-7	2	3	-4
43	4	3	3	0	14	8	10	-9	-4	9	93	-1	-4	4	10	2	-11	2	5	-4	5
44	9	9	5	4	13	6	6	1	6	17	94	2	0	2	2	1	-4	10	0	-9	4
45	2	5	7	1	12	11	3	14	6	9	95	2	5	6	2	-9	3	-4	0	5	4
46	3	4	0	6	7	10	5	3	2	12	96	3	7	0	-1	7	1	-4	0	-3	-5
47	2	12	9	-1	8	-1	10	4	4	6	97	3	1	-1	2	5	1	-1	1	3	0
48	13	9	4	-5	1	6	18	11	-3	-3	98	3	3	-10	-9	0	6	3	-3	-4	0
49	8	1	15	2	-9	3	6	15	8	5	99	-3	6	0	5	-4	7	5	6	-6	2
50	5	10	8	6	8	4	11	4	4	7	100	2	-5	4	7	0	-1	-1	0	1	-3

the specification limits. A check of the normal curve tables showed that about 0.7% of the parts were being machined above the upper specification limit and 6.2% below the lower specification limit giving a total of about 6.9% of the product not meeting the specification. However, due to many factors, it was felt that a loss of 6.9% of the machined parts would be more desirable than to try to rebuild the machine or to do the other things that would be required to improve the 93.1% of acceptable product. Since the production process was in a state of statistical control and since the 93.1% of product conforming to specifications was economically feasible, the control limits on $\bar{x}$ and R were extended for use in future production. The production of the remaining 750 parts as shown in Table 1.2 was begun.

TABLE 1.3.
Table of Random Numbers

62460	89704	36907	37007
48847	77239	48401	05525
16550	24079	98408	35102
08832	44073	12126	27641
28721	70081	86976	35169
50863	54791	00400	65636
58097	03891	28562	25868
22869	46279	29862	87064
18563	40651	18068	18208
90618	11084	35971	67337
15660	13339	27032	00623
64012	63206	51369	77424
76711	94417	43146	31474
30291	25245	71494	58054
07656	06455	21924	36587
71199	30717	49483	41188
94860	94843	63338	63014
13339	59789	34252	76410
15766	10274	32031	09487
73975	64463	97430	36940
48174	75049	48871	21510
44385	41564	97724	07591
08037	69444	60309	01141
74586	00593	75232	88980
43579	08844	49923	72662

The sampling, plotting points, and the calculations of the control limits for the first 25 samples of 3 are shown in Table 1.4 and Fig. 1.4. It is now suggested that the reader try his hand in control chart use and analysis by performing the following exercises:

1) Verify the calculated result for the center lines and control limits for both $\bar{x}$ and R for the first 25 samples as shown in Table 1.4.
2) Extend the control limits on both $\bar{x}$ and R for future production and then continue sampling 3 observations from the 26 th to the 40 th half hour's production and plotting $\bar{x}$ and R. Note: Use the random number table to draw your samples by, say, using the two digits of your age as the row and column of your starting point in the table. Then move in groups of 3 in any direction you desire to

obtain 3 random numbers for each sample of 3 you desire to draw. Omit any duplicated numbers in a sample.

TABLE 1.4.

Data for $\bar{x}$ and R Chart for First 25 Samples from Machined Part Process

Sample Number	Random Number	x_1	x_2	x_3	Total	$\bar{x}$	R
1	493	0	0	−3	−3	−1.0	3
2	785	−2	0	0	−2	−0.7	2
3	280	−1	−11	−10	−22	−7.3	10
4	278	1	−9	−10	−18	−8.0	11
5	627	−4	0	0	−4	−1.3	4
6	907	−9	0	7	−2	−0.7	16
7	584	−9	−5	−9	−23	−7.7	4
8	782	3	−9	−3	−9	−3.0	12
9	214	−1	−1	−4	−6	−2.0	3
10	704	−2	−6	−8	−16	−5.3	6
11	185	−5	−1	−1	−7	−2.3	4
12	198	2	−3	3	2	−0.7	6
13	781	−2	2	−5	−5	−1.7	7
14	945	5	−2	2	5	1.7	7
15	869	−2	1	−3	−4	−1.3	4
16	140	−2	−2	−5	−9	−3.0	3
17	542	−8	−1	4	−5	−1.7	12
18	948	−7	−5	−5	−17	−5.7	2
19	091	1	−3	−7	−9	−3.0	8
20	539	−4	−4	5	−3	−1.0	9
21	463	2	0	−3	−1	−0.3	5
22	178	−6	−1	−4	−11	−3.7	2
23	480	−7	−5	−4	−16	−5.3	3
24	386	−4	0	−1	−5	−1.7	4
25	056	−6	6	3	3	1.0	12
					−187		159

$$\bar{\bar{x}} = 2.49\ , \quad UCL_{\bar{x}} = 4.00\ , \quad LCL_{\bar{x}} = 8.98$$

$$\bar{R} = 6.36\ , \quad UCL_R = 16.3\ , \quad LCL_R = 0$$

3) Continue your sampling and plotting of $\bar{x}$ and R for the production during the 41st to the 50th half hour's production. At each point explain any noticeable change in the machining process.

4) Continue your sampling and plotting of $\bar{x}$ and R for the production during the 51st to the 60th half hour's production. At each point explain any noticeable change in the machining process.

5) Continue your sampling and plotting of $\bar{x}$ and R for the production during the 61st to the 70th half hour's production. At each point explain any noticeable change in the machining process.

6) Beginning with points number 71, use the remaining 30 half hour's production to find new center lines and control limits for the

production process. Make a comparison of current status of the process with that of the original process as reflected by the first 25 samples.

This work should prove to be of considerable interest to the reader. In the next article in this series, Mr. Gold will discuss these six exercises.

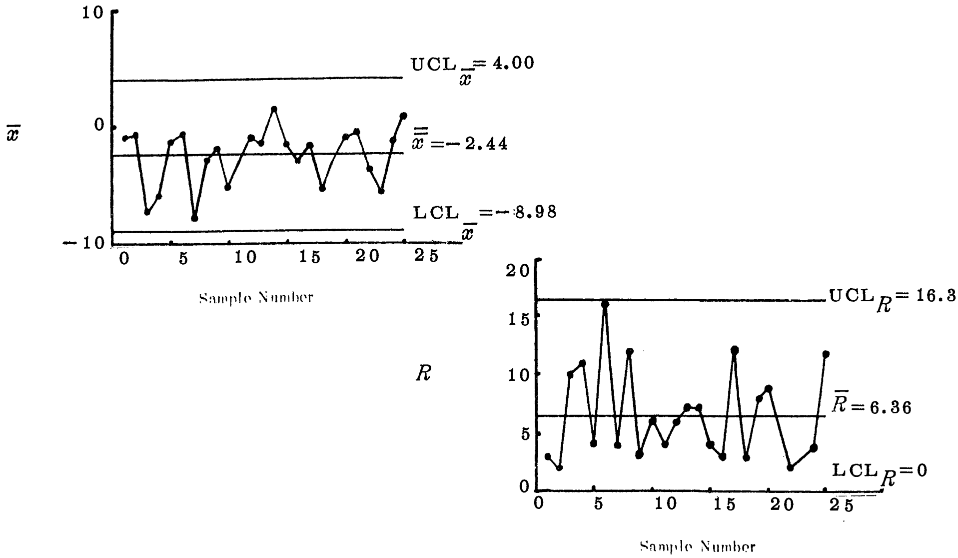

Fig. 1.4. $\bar{x}$ and R Control Charts for First 25 Samples of Machining Process

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A NEW PROBABILITY MODEL FOR BERTRAND'S PARADOX

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Only when the experimental procedure involved is precisely defined can probability theory hope to provide clear, unequivocal answers to simple sounding questions like, "What is the probability that a chord drawn at random inside a circle will be longer than the side of an equilateral triangle inscribed in the circle?" Three models producing results of 1/4, 1/3, and 1/2 are customarily given for this problem, often called Bertrand's Paradox. I shall first list these three, and then give my own which I believe to be new.

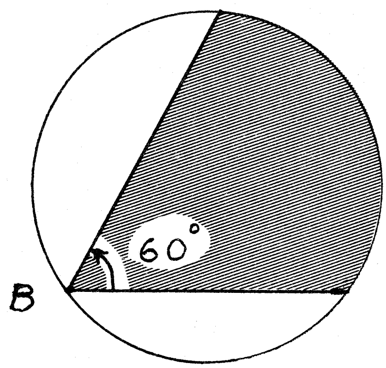
The first model considers the midpoint of the chord to range uniformly over the entire space within the circle of radius r. In this situation only chords whose midpoints are within $r/2$ of the center will be longer than the side of an inscribed equilateral triangle. This model gives a result of 1/4.

The second model chooses a point B on the circle and considers the angle at B between the tangent and the chord uniformly distributed between 0° and 180°. This model gives a result of 1/3.

The third model considers the distance of the midpoint of the chord from the center to be uniformly distributed from 0 (the center of the circle) to r (the length of the radius of the circle). This model gives a result of 1/2.

I now give my conjecture for a model. I postulate that the subject, after choosing the point B on the circle, picks another point, P, *in or on* the circle through which he will draw his chord. If P is chosen inside the shaded area the chord exceeds $\sqrt{3}\,r$ (the side of inscribed equilateral triangle), otherwise not This gives

$$\text{Prob} = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} = .609 \text{ approx.}$$



Which of the four answers is correct? Each is in reference to a certain mechanical procedure for drawing a chord at random. The problem has no solution until the meaning of "draw a chord at random" is made precisely a description of the procedure to be followed. Which of the four is "more nearly true" could perhaps be partially resolved by a psychological "study of the human organism as a random mechanism."

THE NUMBER SYSTEM IN MORE GENERAL SCALES

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One of the reasons for the popularity of our usual scale of notation with base 10 is that it allows the representation of any positive integer n in one and only one way as

$$(1) \qquad n = \sum_{i=0}^{p} c_i 10^i ,$$

where $0 \leqq c_i \leqq 9$, $c_p \geqq 1$.

Similarly using any other base, say ρ, it is well known that any positive integer n can be represented uniquely as

$$(2) \qquad n = \sum_{i=0}^{p} c_i \rho^i ,$$

where $0 \leqq c_i \leqq \rho - 1$, $c_p \geqq 1$.

In the representation (2), our number system can be thought of as being built up by means of a basic set of "building blocks" which we shall designate $f_0, f_1, f_2, f_3, \cdots$, where $f_0 = 1$, $f_1 = \rho$, $f_2 = \rho^2$, $\cdots$.

It is the object of this paper to consider arbitrary nondecreasing sequences of integers $\{f_i\}_{i=0}^{\infty}$, where the f_i are not necessarily distinct, and determine for what sets of this type it is possible to represent every positive integer n as

$$(3) \qquad n = \sum_{i=0}^{\infty} c_i f_i ,$$

where each nonnegative integer coefficient c_i is less than some given number, say k_i. We do not necessarily require here that the representation (3) is unique.

Specifically we shall try to answer the following questions:

1. Is it possible to find necessary and sufficient conditions for the numbers $f_0, f_1, f_2, \cdots$ so that every positive integer can be represented in the form given by (3) for a given set $k_0, k_1, k_2, \cdots$ representing the upper bounds, respectively, for $c_0, c_1, c_2, \cdots$?

2. For what sets $\{f_i\}_{i=0}^{\infty}$ is the representation given by (3) unique for every positive integer n? We know, of course, that the representation is unique if the f_i are the consecutive powers of a number ρ, i.e., $f_i = \rho^i$; it is, therefore, only of interest to determine whether there are any additional

sets $\{f_i\}_{i=0}^{\infty}$ for which the representation is unique.

The answers to these questions are furnished by the following two theorems:

THEOREM 1. Let $\{f_i\}_{i=0}^{\infty}$ be a nondecreasing sequence of positive integers with $f_0 = 1$; then every positive integer n can be represented in the form

$$n = \sum_{i=0}^{\infty} c_i f_i$$

with $0 \leqq c_i \leqq k_i$ for a given set $\{k_i\}_{i=0}^{\infty}$, if and only if

$$f_{p+1} \leqq 1 + \sum_{i=0}^{p} k_i f_i \tag{4}$$

for $p = 0, 1, 2, \cdots$.

THEOREM 2. For a given set $\{k_i\}_{i=0}^{\infty}$ representing the upper bounds for the c_i, there is only one nondecreasing sequence of positive integers $\{f_i\}_{i=0}^{\infty}$ for which the representation of every positive integer n given by (3) with $0 \leqq c_i \leqq k_i$ is unique, namely the set $\{\phi_i\}_{i=0}^{\infty}$, where

$$\begin{aligned} \phi_0 = 1, \quad \phi_1 = 1+k_0, \quad \phi_2 = (1+k_0)(1+k_1), \quad \cdots, \\ \phi_i = (1+k_0)(1+k_1)\cdots(1+k_{i-1}), \quad \cdots. \end{aligned} \tag{5}$$

In answer to question 2, therefore, there are additional sets $\{f_i\}_{i=0}^{\infty}$ for which the representation is unique provided we allow a different upper bound k_i to be prescribed for each coefficient c_i. If, however, we demand that all c_i are bounded by the same constant, say k, i.e., $c_i \leqq k$, for $i = 0, 1, 2, \cdots$, then it follows from theorem 2 that the only set $\{\phi_i\}_{i=0}^{\infty}$ for which the representation of every positive integer n is unique is the one where $\phi_i = (1+k)^i$, which means that the ϕ_i are consecutive powers of the number $1+k$ which is the ρ of representation (2).

It is of interest to note that the prescribed k_i do not have to form a nondecreasing sequence.

A direct consequence of theorem 1 is that, if the set $\{f_i\}_{i=0}^{\infty}$ represents any nondecreasing sequence of positive integers such that every positive integer n can be represented in the form given by (3) with $0 \leqq c_i \leqq k_i$ for a given set $\{k_i\}_{i=0}^{\infty}$, then $f_i \leqq \phi_i$ for $i = 0, 1, 2, \cdots$, where the ϕ_i are defined by (5).

This follows by induction. Since f_0 always has to be 1, the state- is obviously true for $i = 1$. Assuming it now for $i = m$, we have from theorem 1

$$f_{m+1} \leqq 1+\sum_{i=0}^{m} k_i f_i \leqq 1+\sum_{i=0}^{m} k_i \phi_i = \phi_1 + \sum_{i=1}^{m} k_i(1+k_0)(1+k_1)(1+k_2)\cdots(1+k_{i-1})$$

$$= (1+k_0)(1+k_1)(1+k_2)\cdots(1+k_m) = \phi_{m+1} ,$$

which verifies the above statement.

The special case of representation (3) where $0 \leqq c_i \leqq 1$, i. e., where each c_i is either 0 or 1, was considered by Hoggatt and King [1] who defined a sequence of nondecreasing positive integers $\{f_i\}_{i=0}^{\infty}$ to be *complete* if and only if every positive integer n can be represented in the form given by (3) with $0 \leqq c_i \leqq 1$. Brown [2] proved theorems 1 and 2 for the special case of complete sequences of integers.

The present paper might therefore be considered as the extension of Brown's work to a more general class of integers which we might denote by semi-complete integers as given by the following definition.

DEFINITION. A nondecreasing sequence of positive integers $\{f_i\}_{i=0}^{\infty}$ is called *semi-complete* if and only if every positive integer can be represented in the form

$$n = \sum_{i=0}^{\infty} c_i f_i ,$$

where the c_i are nonnegative integers and $c_i \leqq k_i$ for a given set of integers $\{k_i\}_{i=0}^{\infty}$.

Clearly the set of all complete sequences of integers forms a subset of the set of all semi-complete sequences of integers.

By use of the definition above, theorems 1 and 2 can be given much more concise formulations which shall not be stated here since they follow immediately. The definition, however, will be used in the proof of the two theorems which follow.

Proof of theorem 1. We shall first prove that the condition given by (4) is sufficient. We use induction to show that, for any n in the range

$$0 < n < 1+\sum_{i=0}^{m} k_i f_i ,$$

there exists a set $\{c_i\}_{i=0}^{m}$ such that

$$n = \sum_{i=0}^{m} c_i f_i$$

with $0 \leqq c_i \leqq k_i$. This is obviously true for $m = 1$. Now let us assume it holds for $m = M$; then we need to show that for any n in the range

$$0 < n < 1 + \sum_{i=0}^{M+1} k_i f_i$$

there exists a set $\{d_i\}_{i=0}^{M+1}$ with $0 \leqq d_i \leqq k_i$ such that

$$n = \sum_{i=0}^{M+1} d_i f_i .$$

We can restrict our attention to those integers n which are in the range

(6) $$1 + \sum_{i=0}^{M} k_i f_i \leqq n < 1 + \sum_{i=0}^{M+1} k_i f_i ,$$

since the induction hypothesis guarantees the desired representation of n for the range

$$0 < n < 1 + \sum_{i=1}^{M} k_i f_i .$$

From (6) it follows that

$$n - f_{M+1} \geqq 1 + \sum_{i=0}^{M} k_i f_i - f_{M+1} \geqq 0 .$$

If $n - f_{M+1} = 0$, the desired conclusion follows; otherwise we can conclude from

$$0 < n - f_{M+1} \leqq 1 + \sum_{i=0}^{M+1} k_i f_i - f_{M+1} = 1 + \sum_{i=0}^{M} k_i f_i + (k_{M+1} - 1) f_{M+1}$$

that there exists a set $\{c_i\}_{i=0}^{M}$ such that

$$n - f_{M+1} = \sum_{i=0}^{M} c_i f_i + (k_{M+1} - 1) f_{M+1}$$

and

$$n = \sum_{i=0}^{M} c_i f_i + k_{M+1} f_{M+1} = \sum_{i=0}^{M+1} d_i f_i ,$$

where $d_i = c_i$ for $i = 0, 1, 2, \cdots, M$, and $d_{M+1} = k_{M+1}$, so that, as desired, the existence of the set $\{d_i\}_{i=0}^{M+1}$ with $0 \leqq d_i \leqq k_i$ has been established.

To prove that the condition given by (4) is also necessary, we assume there exists a semi-complete sequence of integers $\{f_i\}_{i=0}^{\infty}$ and a $q \geqq 1$ for which (4) is false, i. e.,

$$f_{q+1} > 1 + \sum_{i=0}^{q} k_i f_i . \tag{7}$$

Then

$$f_{q+1} > f_{q+1} - 1 > \sum_{i=0}^{q} k_i f_i .$$

Now let us represent the positive integer $f_{q+1} - 1$ as

$$f_{q+1} - 1 = \sum_{i=0}^{\infty} c_i f_i .$$

Clearly all c_i for $i \geqq q+1$ must be 0 since the left hand side is less than f_{q+1}. Now using as coefficients c_i for $i \leqq q$ the maximum permissible values, i. e., $c_i = k_i$, we obtain

$$\sum_{i=0}^{q} k_i f_i < f_{q+1} - 1 ,$$

the inequality following from (7). This shows that the integer $f_{q+1} - 1$ cannot be represented in the form given by (3) which is a contradiction to our hypothesis that the sequence of integers $\{f_i\}_{i=0}^{\infty}$ was semi-complete.

Proof of theorem 2. We shall first prove that every positive integer n can be represented uniquely as

$$n = \sum_{i=0}^{\infty} c_i \phi_i ,$$

where $0 \leqq c_i \leqq k_i$ for a given set $\{k_i\}_{i=0}^{\infty}$ and where the ϕ_i are given by (5).

This proof will also give a simple procedure for determining the c_i.

Dividing n by $1+k_0$ there exists a unique representation

(8) $$n = n_1(1+k_0)+c_0\ , \quad \text{where } 0 \leqq c_0 \leqq k_0 \text{ and } n_1 < n\ .$$

If $n_1 \geqq 1+k_1$, we divide n_1 by $1+k_1$ and obtain

(9) $$n_1 = n_2(1+k_1)+c_1\ , \quad \text{where } 0 \leqq c_1 \leqq k_1 \text{ and } n_2 < n_1\ .$$

If $n_2 \geqq 1+k_2$, we divide n_2 by $1+k_2$ and obtain

(10) $$n_2 = n_3(1+k_2)+c_2\ , \quad \text{where } 0 \leqq c_2 \leqq k_2 \text{ and } n_3 < n_2\ .$$

Since the sequence $n, n_1, n_2, \cdots$ is decreasing, there must be some number, say n_p, in the series for which $n_p < 1+k_p$. Letting $n_p = c_p$, and eliminating $n_1, n_2, \cdots, n_p$ successively from the system of equations (8), (9), (10), $\cdots$ and ending with

$$n_{p-1} = n_p(1+k_{p-1})+c_{p-1}\ , \quad n_p = c_p\ ,$$

we obtain

$$n = c_0 + c_1(1+k_0)+c_2(1+k_0)(1+k_1)+\cdots+c_p(1+k_0)(1+k_1)\cdots(1+k_{p-1})$$

$$= \sum_{i=0}^{p} c_i\phi_i\ ,$$

where ϕ_i is given as defined by (5).

To prove that for a given set $\{k_i\}_{i=0}^{\infty}$, the set $\{\phi_i\}_{i=0}^{\infty}$ is the only semi-complete set for which the representation of every positive integer n by means of (3) is unique, we assume that some integer n can be represented uniquely both by a set $\{f_i\}_{i=0}^{\infty}$ and by $\{\phi_i\}_{i=0}^{\infty}$ with ϕ_i as defined by (5), i. e.,

$$n = \sum_{i=0}^{\infty} c_i f_i = \sum_{i=0}^{\infty} d_i\phi_i\ .$$

If these two representations are different, there must be a least number $m \geqq 1$ (since $f_0 = \phi_0 = 1$) for which $f_m \neq \phi_m$. From theorem 1 it follows that

$$f_m \leqq 1+\sum_{i=0}^{m-1} k_i f_i = 1+\sum_{i=0}^{m-1} k_i\phi_i = \phi_m\ .$$

Since $f_m \neq \phi_m$, we can conclude that

$$f_m < 1 + \sum_{i=0}^{m-1} k_i f_i$$

and, using the statement in the second sentence of the proof of theorem 1, we know that the number f_m can be represented as

$$f_m = \sum_{i=0}^{m-1} c_i f_i$$

with $0 \leqq c_i \leqq k_i$. But f_m can also be represented by the integer f_m itself and thus has two representations which shows that the set $\{f_i\}_{i=0}^{\infty}$ is not one by means of which all positive integers can be represented uniquely.

It is also possible to generalize some of the other theorems which Brown [2] proved for the special case where all k_i are equal to 1. Thus, theorem 3 answers the question under what conditions the semi-completeness of a sequence is preserved if one of the numbers of the sequence except the first is removed.

THEOREM 3. Let $\{f_i\}_{i=0}^{\infty}$ be a nondecreasing semi-complete sequence of positive integers; then a necessary and sufficient condition for the sequence to remain semi-complete after the removal of any number except the first from the sequence is

$$f_{p+1} \leqq 1 + \sum_{i=0}^{p-1} k_i f_i$$

for $i = 1, 2, 3, \cdots$.

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ELEMENTARY OBSERVATIONS CONCERNING EULER'S PRIME GENERATING POLYNOMIAL $f(n) = n^2 - n + 41$

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In 1772 Euler discovered that the polynomial $f(n) = n^2 - n + 41$ generates 40 different prime numbers for $n = 1, 2, \cdots, 40$. [1] This function has an amazing ability to generate primes even beyond $n = 40$. Using an IBM 650 Digital Computer, it was found that between $n = 1$ and $n = 2{,}398$ inclusive, precisely one-half of the $f(n)$ are prime integers. Another observation of interest is that $g(n) = f(-3n + 82) = 9n^2 - 489n + 6{,}683$ also yields 40 different prime numbers for $n = 1, 2, \cdots, 40$.

$f(n)$ is never divisible by a prime less than 41. To prove this, it is sufficient to show that $f(n) \not\equiv 0 \bmod p$ for all $p < 41$ and all $1 < n \leqq p$. But $f(n)$ is a prime number greater than p for all $n < 41$. In this connection, it is of interest to note that D. H. Lehmer's polynomial $n^2 - n + 67{,}374{,}467$ is never divisible by any prime less than 107. [2]

$f(n)$ is never a perfect square except for $n = 41$. The proof is as follows: For $n > 41$, $f(n+1) > n^2 > f(n)$. For $n < 41$, $f(n)$ is a prime number. For $n = 41$, $f(n) = 41^2$.

There are infinitely many composite $f(n)$ since $f(n^2 + 41) = f(n)f(n+1)$ for all n.

The smallest $f(n)$ which is divisible by three not necessarily distinct prime factors is $f(421)$. This result was found by laborious calculation with a desk computer. The smallest $f(n)$ which is divisible by four not necessarily distinct prime factors is $f(1722)$. This latter result can be verified in an elegant manner as follows. Since $f(n)$ is never divisible by a prime less than 41, the smallest $f(n)$ containing four factors should be at least equal to 41^4. But this is a perfect square. The next smallest $f(n)$ with four factors is $41^3 \cdot 43$. Now $f(41) = 41^2$ and $f(42) = 41 \cdot 43$ so that $f(41)f(42) = f(41^2 + 41) = f(1722) = 41^3 \cdot 43$.

If $h(y)$ is defined as the smallest value of n for which $f(n)$ has at least y not necessarily distinct prime factors, then the above results show that $h(3) = 421$ and $h(4) = 1722$. Since $f(n)$ is never divisible by a prime less than 41, then $h^2(y) > f[h(y)] \geqq 41^y$ for $y > 2$. Thus for odd $y > 2$, $h(y) > 41^{y/2}$, but since $f(n)$ cannot be a perfect square, $h(y) > 41^{(y-1)/2}\, 43^{1/2}$ for even y. Noting that $f(139{,}564) = 41^4(61)(113)$ contains six factors it may be concluded that $139{,}564 \geqq h(5) \geqq 10{,}764$ and that $139{,}564 \geqq h(6) \geqq 70{,}582$.

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π_t: 1832 - 1879

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When mathematicians are thought of, who remembers James Smith? Or Daniel West? Few people indeed. Yet these men (and 42 others) performed a valuable service in the middle of the last century: they kept track of π_t, the ratio of the circumference of a circle to its diameter at time t. See the table for the results of their calculations, rounded off to five decimal places. The data are mostly from DeMorgan [1] and Gould [2]. Lately, very little has been done in this field; we have let π_t get away from us.

t	π_t	Calculator	t	π_t	Calculator
1832	3.06250	Parsey	1862	3.14159	Benson
1833	3.20222	Baddeley	1862	3.14214	Houlston
1833	3.16483	Bouche	1862	3.20000	Pratt
1835	3.20000	Oliveira	1863	3.14063	Dean
1836	3.12500	Lacomme	1865	3.16049	Faber
1837	3.23077	Bennett	1866	3.24000	May
1841	3.12019	McCook	1868	3.14214	Grosvenor
1843	3.04862	Johnson	1868	3.14159	Harbord
1844	3.17778	Dennison	1869	3.12500	Dircks
1845	3.16667	Davis	1871	3.15470	G. W. B.
1846	3.17480	Young	1871	3.15544	Terry
1848	3.20000	Peters	1872	3.16667	"A. Finality'
1848	3.12500	Merceron	1873	3.14286	Myers
1849	3.14159	deGelder	1874	3.15208	Brower
1850	3.14159	Parker	1874	3.14270	Harris
1851	3.14286	Adorno	1874	3.15300	Stacy
1853	3.12381	"Futurus"	1875	3.14270	Goodsell
1854	3.17124	Bouche	1875	3.15333	Weatherby
1855	3.15532	Smith, A.	1876	3.13397	Cart
1858	3.20000	Anghera	1878	3.20000	"Durham"
1859	3.14159	Gee	1878	3.13514	Gidney
1860	3.12500	Smith, J.	1879	3.14286	Crabb
1860	3.14241	Hailes			

But perhaps something can be saved. If we construct a least-squares line using the data of the table, we find that $\pi_t = .0000056060t + 3.14281$ where t is measured in years A.D.; in particular, $\pi_{1962} = 3.15381$. Consider the implications of our relation. For one thing, we see that the Biblical value $\pi_t = 3$ was an excellent approximation for those days. For another, schoolchildren in 10201 can look forward with more pleasure than usual to June, for in that month, $\pi_t = 3.20000$, and their calculations will be much simpler.

There are discrepancies, though. We find that π_t was 3.1415926535⋯ sometime around 10:54 P.M. on November 10, 219 B.C. (using the

Gregorian calendar and Greenwich time). What happened then that fixed this erroneous approximation as an exact value? History is silent. Further, our expression gives the date of the creation – when π_t was zero – as 560,615 B. C., agreeing neither with astronomical theory nor with Archbishop Ussher's chronology. Clearly, more research is needed.

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TO THE PLANE

Thou art the madhouse mirror of
Our geometric thought.
Unhampered, theoretically,
We project higher space on thee
Or sketch, topologetically,
The hardy facts we've wrought.

Marlow Sholander

n AND $n+1$ CONSECUTIVE INTEGERS WITH EQUAL SUMS OF SQUARES

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The following development may be considered to stem from the relation:

$$5^2 = 3^2 + 4^2$$

in which one consecutive number (if we may speak of such) has a square equal to the sum of the squares of two consecutive numbers. Additional instances of this type are:

$$29^2 = 20^2 + 21^2$$

$$169^2 = 119^2 + 120^2$$

and many others forming an infinite sequence.

The problem is to determine, if possible, two consecutive integers the sum of whose squares equals the sum of the squares of three consecutive integers; three consecutive integers, the sum of whose squares equals the sum of the squares of four consecutive integers; and so on.

Preliminary investigations for specific cases yielded the following results which are set down as they played a role in providing a lead or two in the general case. The number in the first column indicates the first in a sequence and likewise the number in the second column.

Table I

1 - - 2		2 - - 3		3 - - 4	
5	3	13	10	25	21
29	20	133	108	361	312
169	119	1321	1078		

Thus, from the 2 - -3 table, we would know that:

$$133^2 + 134^2 = 108^2 + 109^2 + 110^2 .$$

These skirmishings with individual instances pointed the way to the following analysis for the case of n and $n+1$ consecutive integers. It is not difficult to show that the sum of the squares of n consecutive integers:

$$x^2 + (x+1)^2 + (x+2)^2 + \cdots + (x+n-1)^2$$

may be summarized in the form:

$$nx^2 + n(n-1)x + \frac{(n-1)n(2n-1)}{6} .$$

Thus, if x be the first of n consecutive integers and y the first of $n+1$ consecutive integers, the condition that they have equal sums of squares is:

(1) $$nx^2 + n(n-1)x + \frac{(n-1)n(2n-1)}{6} = (n+1)y^2 + (n+1)ny + \frac{n(n+1)(2n+1)}{6}$$

which leads to:

(2) $$nx^2 + n(n-1)x - n^2 = (n+1)y^2 + n(n+1)y \ .$$

We shall let $y = ny'$. (This is not necessary in all instances, but can be done inasmuch as the left-hand side is divisible by n.) This substitution leads to the equation:

(3) $$(x+n)(x-1) = n(n+1)y'(y'+1) \ .$$

Setting $z = x-1$, (3) becomes:

(4) $$z(z+n+1) = n(n+1)y'(y'+1) \ .$$

By making the substitution $z = (n+1)z'$, this reduces to:

(5) $$(n+1)z'(z'+1) = ny'(y'+1) \ .$$

Finally, let $z' = m$ and $y' = m + \alpha$. This gives:

$$(n+1)m(m+1) = n(m+\alpha)(m+\alpha+1)$$

or

(6) $$m^2 + m(1-2n\alpha) - n\alpha(\alpha+1) = 0 \ .$$

If r_1 and $-r_2$ are the roots of this equation, then

(7) $$r_1 - r_2 = 2n\alpha - 1$$

and

(8) $$r_1 r_2 = n\alpha(\alpha+1) \ .$$

From these relations:

$$(r_1 - r_2)^2 + 4r_1 r_2 = (2n\alpha - 1)^2 + 4n\alpha(\alpha+1)$$

or

(9) $$(r_1 + r_2)^2 = 4n(n+1)\alpha^2 + 1 \ .$$

Solutions will be obtained for integral values of α which yield an integral perfect square, for then we shall be able to find integral values of r_1 and r_2 which in turn will lead to integral values of z' and y' and eventually of x and y.

First family of solutions. Now, it is immediately evident that a solution is obtained for $\alpha = 1$. This makes

$$(r_1 + r_2)^2 = 4n^2 + 4n + 1 = (2n+1)^2$$

so that

$$r_1 + r_2 = 2n+1$$

$$r_1 - r_2 = 2n-1 \qquad \text{by (7)} \ .$$

Thus

$$z' = m = r_1 = 2n\ , \quad x = z+1 = 2n(n+1)+1 = 2n^2+2n+1\ ,$$

$$y' = r_1 + \alpha = 2n+1 \quad \text{and} \quad y = ny' = n(2n+1) = 2n^2+n\ .$$

Thus we have an infinity of solutions corresponding to $\alpha = 1$. This we shall refer to as series 1 in view of the fact that other values of α will be obtained in the sequel to provide additional series. The following table may be derived from the above formulae.

TABLE II

n	x	y
1	5	3
2	13	10
3	25	21
4	41	36
5	61	55
6	85	78
7	113	105
8	145	136
9	181	171
10	221	210
11	265	253

To point up the meaning of this table, consider the case $n = 9$. We should have:

$$\sum_{k=181}^{189} k^2 = \sum_{k=171}^{180} k^2\ .$$

Now the common value of these two sums is found to be 308,085. Note that the values in the table for 1, 2 and 3 correspond respectively to the first values in the tables previously given for $1--2$, $2--3$ and $3--4$ respectively. (See Table I.)

Second series. To arrive at a second series, we take the next values of α for $n = 1$, $n = 2$, and $n = 3$, namely $\alpha = 6$, $\alpha = 10$, and $\alpha = 14$ respectively, corresponding to the solutions (29, 20), (133, 108), and (361, 312) found in the second line of Table I. These numerical results were obtained in the preliminary investigations mentioned at the beginning of this article. From the fact that there are equal differences among the values of α, one might suspect that it would be a linear function of the form $an+b$. If so, it would have to be $4n+2$. Trying this out in the relation:

$$(r_1+r_2)^2 = 4n(n+1)\alpha^2+1$$

we find that $(r_1+r_2)^2$ is indeed a perfect square $(8n^2+8n+1)^2$. This provides us with a second series, but we shall not linger here at the moment to work out specific values.

Additional series. By this time, it might be suspected that there would be other values of α of successively higher degree in n. If we take our relation:

$$(r_1 + r_2)^2 = 4n(n+1)\alpha^2 + 1$$

and represent it as:

$$q_\lambda^2 = 4n(n+1)\alpha_\lambda^2 + 1$$

where λ indicates the series in question, we would have thus far:

$$q_1 = 2n+1 , \quad \alpha_1 = 1$$

$$q_2 = 8n^2 + 8n + 1 , \quad \alpha_2 = 4n + 2 .$$

By the use of analogy, trial and error, and incipient recursion formulas, additional expressions were obtained as follows:

$$q_3 = 32n^3 + 48n^2 + 18n + 1 , \quad \alpha_3 = 16n^2 + 16n + 3$$

$$q_4 = 128n^4 + 256n^3 + 160n^2 + 32n + 1 , \quad \alpha_4 = 64n^3 + 96n^2 + 40n + 4$$

$$q_5 = 512n^5 + 1280n^4 + 1120n^3 + 400n^2 + 50n + 1 ,$$

$$\alpha_5 = 256n^4 + 512n^3 + 336n^2 + 80n + 5 .$$

At this point, the following recursion formulas were established:

$$(10) \qquad \begin{aligned} q_\lambda &= (4n+2)q_{\lambda-1} - q_{\lambda-2} \\ \alpha_\lambda &= (4n+2)\alpha_{\lambda-1} - \alpha_{\lambda-2} \end{aligned}$$

provided q_0 is taken as 1 and α_0 is assumed to be 0.

To show that there is indeed an infinite sequence of families of solutions, it remains to prove that these recursion formulas hold in general.

Proof of the recursion formulas. Let us suppose that the following relations apply up to $\lambda - 1$ inclusive:

$$(11) \qquad q_r = (4n+2)q_{r-1} - q_{r-2} ; \quad \alpha_r = (4n+2)\alpha_{r-1} - \alpha_{r-2} .$$

Also that

$$(12) \qquad q_r^2 = 4n(n+1)\alpha_r^2 + 1 .$$

We are to show that if

$$(13) \qquad q_\lambda = (4n+2)q_{\lambda-1} - q_{\lambda-2}$$

and

$$(14) \qquad \alpha_\lambda = (4n+2)\alpha_{\lambda-1} - \alpha_{\lambda-2}$$

then they are also related by the formula

$$q_\lambda^2 = 4n(n+1)\alpha_\lambda^2 + 1 .$$

By (13),

(15) $$q_\lambda^2 = (4n+2)^2 q_{\lambda-1}^2 - 2(4n+2)q_{\lambda-1}q_{\lambda-2} + q_{\lambda-2}^2 .$$

Substituting from (12),

(16) $$q_\lambda^2 = 4n(n+1)(4n+2)^2\alpha_{\lambda-1}^2 + (4n+2)^2 + 4n(n+1)\alpha_{\lambda-2}^2 + 1 - 2(4n+2)q_{\lambda-1}q_{\lambda-2} .$$

By (14),

(17) $$4n(n+1)\alpha_\lambda^2 + 1 = 4n(n+1)(4n+2)^2\alpha_{\lambda-1}^2 - 8n(n+1)(4n+2)\alpha_{\lambda-1}\alpha_{\lambda-2} + 4n(n+1)\alpha_{\lambda-2}^2 + 1 .$$

Comparing (16) and (17) which should be equal if we are to prove the recursion formula, we see that it remains to be demonstrated that:

(18) $$(4n+2)^2 - 2(4n+2)q_{\lambda-1}q_{\lambda-2} = -8n(n+1)(4n+2)\alpha_{\lambda-1}\alpha_{\lambda-2} .$$

Proof of relation (18). To arrive at this result, we go back and build up our demonstration by mathematical induction.

For q_1, q_2, α_1, and α_2,

(19) $$(4n+2)^2 - 2(4n+2)q_2q_1 = (4n+2)^2 - 2(4n+2)^2q_1^2 + 2(4n+2)q_1$$

since

$$q_2 = (4n+2)q_1 - 1 .$$

This equals in turn:

(20) $$2(4n+2)^2 - 8n(4n+2)^2(n+1)\alpha_1^2 - 2(4n+2)^2$$

since

$$q_1 = 2n+1 \quad \text{and} \quad q_1^2 = 4n(n+1)\alpha_1^2 + 1 .$$

Finally, this can be put into the form:

(21) $$-8n(n+1)(4n+2)\alpha_1\alpha_2$$

since

$$\alpha_2 = (4n+2)\alpha_1 .$$

Thus the relation holds in this case.

As a second step, it can be demonstrated that:

$$(4n+2)^2 - 2(4n+2)q_3q_2 = -8n(n+1)(4n+2)\alpha_3\alpha_2$$

but inasmuch as the argument is exactly the same as for the general case, this duplication will be avoided. We proceed then to the relation (18). By the substitution

$$q_{\lambda-1} = (4n+2)q_{\lambda-2} - q_{\lambda-3}$$

the left-hand side becomes

(22) $$(4n+2)^2 - 2(4n+2)^2 q^2_{\lambda-2} + 2(4n+2)q_{\lambda-2}q_{\lambda-3}\,.$$

Using relation (12), this becomes

(23) $$-8n(n+1)(4n+2)^2\alpha^2_{\lambda-2} - (4n+2)^2 + 2(4n+2)q_{\lambda-2}q_{\lambda-3}\,.$$

Since in our inductive argument, we assume that (18) holds for all previous values of λ, we can substitute:

$$2(4n+2)q_{\lambda-2}q_{\lambda-3} = (4n+2)^2 + 8n(n+1)(4n+2)\alpha_{\lambda-2}\alpha_{\lambda-3}$$

so that (23) becomes :

(24) $$-8n(n+1)(4n+2)^2\alpha^2_{\lambda-2} + 8n(n+1)(4n+2)\alpha_{\lambda-2}\alpha_{\lambda-3}\,.$$

Finally, the relation

$$\alpha_{\lambda-1} = (4n+2)\alpha_{\lambda-2} - \alpha_{\lambda-3}$$

reduces this to

$$-8n(n+1)(4n+2)\alpha_{\lambda-1}\alpha_{\lambda-2}\,.$$

Therefore, the relation (18) holds in general and the recursion formulas (10) are true for all values of n.

Calculations. It would be possible to obtain generalized formulas for the various series, but this would be a laborious and complicated way of arriving at numerical values. Instead, we can obtain recursion formulas for x and y which will enable us to set up tables of values at once. We shall first derive these results.

Let x_λ be the initial number of the group of n consecutive integers in the λ series, it being understood that x_λ is a function of n. Similarly, let y_λ be the initial number of the group of $n+1$ consecutive integers in the λ series. From previous relations :

$$r_1 - r_2 = 2n\alpha_\lambda - 1$$

$$r_1 + r_2 = q_\lambda\,.$$

Therefore,

$$r_1 = \frac{q_\lambda + 2n\alpha_\lambda - 1}{2}$$

so that

$$x_\lambda = (n+1)r_1 + 1 = (n+1)\left[\frac{q_\lambda + 2n\alpha_\lambda - 1}{2}\right] + 1$$

or

(25) $$x_\lambda = \left(\frac{n+1}{2}\right)q_\lambda + n(n+1)\alpha_\lambda - \left(\frac{n-1}{2}\right).$$

Now

$$q_\lambda = (4n+2)q_{\lambda-1} - q_{\lambda-2}$$
$$\alpha_\lambda = (4n+2)\alpha_{\lambda-1} - \alpha_{\lambda-2}.$$

Therefore,

$$x_\lambda = \left(\frac{n+1}{2}\right)(4n+2)q_{\lambda-1} + n(n+1)(4n+2)\alpha_{\lambda-1} - \left(\frac{n+1}{2}\right)q_{\lambda-2} - n(n+1)\alpha_{\lambda-2} - \left(\frac{n-1}{2}\right)$$

or

(26) $$x_\lambda = (4n+2)x_{\lambda-1} - x_{\lambda-2} + 2n(n-1).$$

Similarly, we calculate a formula for y_λ, arriving at the result:

(27) $$y_\lambda = (4n+2)y_{\lambda-1} - y_{\lambda-2} + 2n^2.$$

Values of x_0 and y_0. The values of x_0 and y_0 need to be determined. We have:

$$x_1 = (n+1)\left[\frac{q_1 + 2n\alpha_1 - 1}{2}\right] + 1$$
$$x_1 = 2n^2 + 2n + 1.$$

Similar calculations lead to:

$$x_2 = 8n^3 + 14n^2 + 6n + 1.$$

From the recursion formula:

$$x_2 = (4n+2)x_1 - x_0 + 2n(n-1)$$

we obtain on substituting the value of x_1

$$x_2 = 8n^3 + 14n^2 + 6n + 2 - x_0.$$

Comparison with the previously obtained formula for x_2 shows that

$$x_0 = 1.$$

Proceeding in like manner, we find that

$$y_0 = 0.$$

Tables of solutions. We are now in a position to set up tables of solutions. This can be done separately for x_λ and y_λ. To show how the tables may be built up, consider the x's for $n = 11$. The first column is obtained from the formula:

$$x_1 = 2n^2 + 2n + 1 \; .$$

TABLE III

Solutions for x_λ, the first number in a series of n consecutive integers

n	x_1	x_2	x_3	x_4	x_5
1	5	29	169	985	5741
2	13	133	1321	13081	129493
3	25	361	5041	70225	978121
4	41	761	13681	245521	4405721
5	61	1381	30361	666601	14634901
6	85	2269	58969	1530985	39746701
7	113	3473	104161	3121441	93539153
8	145	5041	171361	5821345	197754481
9	181	7021	266761	10130041	384674941
10	221	9461	397321	16678201	700087301
11	265	12409	570769	26243185	1206615961
12	313	15913	795601	39764401	1987424713
13	365	20021	1081081	58358665	3150287141
14	421	24781	1437241	83335561	4832025661
15	481	30241	1874881	116212801	7203319201
16	545	36449	2405569	158731585	10473879521
17	613	43453	3041641	212871961	14897996173
18	685	51301	3796201	280868185	20780450101
19	761	60041	4683121	365224081	28482795881
20	841	69721	5717041	468728401	38430012241

Also, $x_0 = 1$. Thus, for $n = 11$, $x_1 = 265$, $x_0 = 1$. Hence

$$x_2 = (4n+2)x_1 - x_0 + 2n(n-1)$$

or

$$x_2 = 46x_1 - x_0 + 220 = 46 \cdot 265 - 1 + 220 = 12409 \; .$$

Then

$$x_3 = 46 \cdot 12409 - 265 + 220 = 570769$$

and so on.

As an indication of the use of these tables, consider $n = 7$ for series 5. Taking the x and y values at corresponding points in the two tables, we should have the relation:

$$\sum_{k=93539153}^{93539159} k^2 = \sum_{k=87497865}^{87497872} k^2 \; .$$

The common value for these two expressions is:

$$61{,}247{,}015{,}936{,}346{,}380\ .$$

TABLE IV

Solutions for y_λ, *the first number in a series of* $n+1$ *consecutive integers.*

n	y_1	y_2	y_3	y_4	y_5
1	3	20	119	696	4059
2	10	108	1078	10680	105730
3	21	312	4365	60816	847077
4	36	680	12236	219600	3940596
5	55	1260	27715	608520	13359775
6	78	2100	54594	1417416	36798294
7	105	3248	97433	2919840	87497865
8	136	4752	161560	5488416	186444712
9	171	6660	253071	9610200	364934691
10	210	9020	378830	15902040	667507050
11	253	11880	546469	25125936	1155246829
12	300	15288	764388	38204400	1909455900
13	351	19292	1041755	56235816	3035692647
14	406	23940	1388506	80509800	4668180286
15	465	29280	1815345	112522560	6974583825
16	528	35360	2333744	153991530	10161107748
17	595	42228	2955943	206874360	14478249835
18	666	49932	3694950	273377016	20226204882
19	741	58520	4564541	355976400	27761595381
20	820	68040	5579260	457432080	37503852100

Summary and conclusion. In the above development, we have shown that for each value of n, there is an infinity of possible solutions such that n and $n+1$ consecutive integers will have equal sums of squares. Moreover, these solutions for the various values of n arrange themselves in an infinity of families. No attempt has been made to determine whether these are the only solutions, a matter that is susceptible of additional study and research.

Note on calculation. For finding the sums of the squares of consecutive integers, we may add the squares directly when they are few in number and not too large. Otherwise, one of the following methods is preferable:

METHOD 1. Since the sum of the squares of the first n consecutive numbers is given by the formula:

$$\frac{n(n+1)(2n+1)}{6}$$

the sum of the squares of the consecutive integers from $k+1$ to r inclusive would be:

$$\frac{r(r+1)(2r+1)}{6} - \frac{k(k+1)(2k+1)}{6} .$$

Example.

$$\sum_{k=3248}^{3255} k^2 = \frac{3255 \cdot 3256 \cdot 6511}{6} - \frac{3247 \cdot 3248 \cdot 6495}{6} = 84{,}578{,}060 .$$

METHOD 2. For larger numbers for which the figures start to get beyond the capacity of one's calculator, a revised formula with all positive quantities enables one to work with somewhat smaller figures. We note that in the numerator we have a difference of the form

$$abc - a'b'c'$$

where $a - a' = b - b' = (c - c')/2$ is the number of squares that are being summed. This difference can be written as

$$abc - a'bc + a'bc - a'b'c + a'b'c - a'b'c'$$

or

$$(a-a')bc + (b-b')a'c + (c-c')a'b'$$

or, remembering that $c - c' = 2(a - a')$, as

$$(a-a')[c(a'+b) + 2a'b'] .$$

Introducing the factor 1/6, the sum of the squares can be represented as

$$(a-a')\ \frac{c(a'+b)+2a'b'}{6} .$$

Example.

$$\sum_{k=1417416}^{1417422} k^2 .$$

Since $a = 1417422$, $b = 1417423$, $c = 2834845$, $a' = 1417415$, the sum equals

$$\frac{7}{6}[2834845 \cdot 2834838 + 2 \cdot 1417415 \cdot 1417416] = 14{,}063{,}536{,}350{,}955 .$$

APPROXIMATING THE ZEROS OF A POLYNOMIAL

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The following method of approximating the zeros of a polynomial does not require any knowledge of differentiation or of graphing and may be presented in a manner comprehensible to anyone who has had an algebra course which includes the topics of partial fractions, quadratic equations, the algebraic identity

$$1-u^m = (1-u)(1+u+\cdots+u^{m-1})$$

from which one has

$$(1') \qquad \frac{1}{1-u} = 1+u+\cdots+\frac{u^m}{1-u} \qquad \text{for } u \neq 1 ,$$

and a rudimentary concept of approach. (Usually, the last two topics appear together in a unit on geometric progressions.) In connection with quadratic equations, one should have dealt with complex numbers.

Starting with the above, identity, induction leads to the further identity

$$(1) \qquad (1-u)^{-k} = 1+ku+\cdots+\frac{k(k+1)\cdots(k+m-2)}{(m-1)!}u^{m-1}+u^m(1-u)^{-k}P_{k,m}(u) ,$$

for $u \neq 1$ and k any natural number, $P_{k,m}(u)$ being a polynomial in u of degree $k-1$.

Summation notation, binomial coefficients, synthetic division, and a rigorous concept of limit, while helpful, need not be used. The first two of these topics enable one to rewrite (1) in the form

$$(1-u)^{-k} = \sum_{i=0}^{m-1} \binom{k+i-1}{i} u^i + u^m(1-u)^{-k}P_{k,m}(u) .$$

Although the development presented here should be of interest in some college freshman courses, the results should be of greater interest in numerical analysis and its applications. The method adapts well to desk calculators, and programming the method for large computers is probably simple. In connection with this use of the method, an alternate viewpoint to that which is most adaptable to an elementary algebra course has also been presented here.

Let $f(x)$ be a polynomial in x of degree n. That is, $f(x) = a_0 \prod_{k=1}^{n} (x-r_k)$ with $a_0 \neq 0$. If $r_i \neq r_j$ for $i \neq j$ and $g(x)$ is any polynomial of degree less than n, then the method of partial fractions applied to the rational function $g(x)/f(x)$ determines a set of constants

$$w_k = \frac{g(r_k)}{a_0 \prod_{i \neq k} (r_k - r_i)}$$

and the expansion

$$\frac{g(x)}{f(x)} = \sum_{k=1}^{n} \frac{w_k}{x - r_k} .$$

Factoring $1/x$ out of the summation and then applying the identity (1′) with $u = r_k/x$ gives

$$\frac{g(x)}{f(x)} = \frac{1}{x} \sum_{k=1}^{n} \frac{w_k}{1 - \frac{r_k}{x}} = \frac{1}{x} \sum_{j=0}^{m-1} \sum_{k=1}^{n} w_k \left(\frac{r_k}{x}\right)^j + \sum_{k=1}^{n} \frac{w_k \left(\frac{r_k}{x}\right)^m}{x - r_k} .$$

Letting $M_j = \sum_{k=1}^{n} w_k r_k^j$, the jth moment, with respect to the w_k's, of the zeros of the polynomial $f(x)$, one has, for each natural number m,

$$\frac{g(x)}{f(x)} = \sum_{j=0}^{m-1} \frac{M_j}{x^{j+1}} + \sum_{k=1}^{n} \frac{w_k r_k^m}{x^m (x - r_k)} . \tag{2}$$

This expansion in negative powers of x can be found by long division or by some synthetic process.* Lanczos has considered this for the case $g(x) = f'(x)$. It is simpler to take $g(x) = a_0$, in which case $w_k \prod_{i \neq k} (r_i - r_k) = 1$.

Generally, if $g(x)$ has no zeros in common with $f(x)$ then $\prod_{k=1}^{n} w_k \neq 0$.

Essentially, (2) gives the Laurent development of $g(x)/f(x)$. The coefficients of the expansion, or moments of the zeros of $f(x)$ with respect to the w_k's are the numbers to be used in approximating the zeros of $f(x)$. The following theorem enables one to find these moments by an alternate method.

THEOREM I: If $f(x) = \sum_{k=0}^{n} a_k x^{n-k}$ with $a_0 a_n \neq 0$ and $r_1, r_2, \cdots, r_m$ is a set of distinct zeros of $f(x)$ and $M_j = \sum_{i=1}^{m} w_i r_i^j$ is the jth moment of this set of zeros with respect to the m-tuple $\{w_i\}_{i=1,\cdots,m}$, then

$$\sum_{k=0}^{n} a_k M_{j-k} = 0 \tag{3}$$

for every integer j.

Knowledge of m consecutive M_j's determines the m-tuple $\{w_i\}$ while

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*Hall and Knight, *Higher Algebra*, p. 434.
Lanczos, *Applied Analysis*, p. 13.

knowledge of n consecutive M_j's determines the complete set of moments.

Proof:

$$\sum_{k=0}^{n} a_k M_{j-k} = \sum_{k=0}^{n} a_k \sum_{i=1}^{m} w_i r_i^{j-k} = \sum_{i=1}^{m} w_i r_i^{j-n} \sum_{k=0}^{n} a_k r_i^{n-k} = 0 .$$

For m consecutive values of M_j, the determinant of the coefficients of the w_i's is a non-zero multiple of a Van der Monde determinant, and hence one can solve for them, since the r_i's are distinct. The rest of the theorem follows from the fact that $a_0 a_n \neq 0$.

Remarks: Relation (3) is called a linear homogeneous difference equation with constant coefficients and $f(x)$ is called its auxilliary polynomial. If $m = n$ and one lets $M_0 = 1$ and $M_j = 0$ for $j = -1, \cdots, -n+1$, then repeated use of (3) enables one to determine the same set of positive moments which occur in (2). The difference equation (3) gives a simple method of determining a set of moments on a calculator.

Now assume that no two zeros of $f(x)$ are the same and that r_1 is greater in absolute value than is any other zero. Since

$$M_j = w_1 r_1^j \left\{ 1 + \sum_{i=2}^{n} \frac{w_i}{w_1} \left(\frac{r_i}{r_1}\right)^j \right\},$$

one has, for large j

$$\text{(4)} \qquad \frac{M_j}{w_1} \approx r_1^j \quad \text{(Root approximation)} .$$

$$\frac{M_{j+1}}{M_j} \approx r_1 \quad \text{(Ratio approximation)} .$$

If r_1 is smaller in absolute value than is any other zero of $f(x)$, then (4) is true for numerically large negative values of j.

Throughout the remainder of this paper, further theoretical development will be interspersed with simple examples to show how this method works.

Example 1. Find the numerically greatest and smallest zeros of

$$x^3 - 5x^2 + 2x + 8 ,$$

approximately.

To determine the numerically greatest zero, one needs to determine a set of positive moments first. Letting $M_{-2} = M_{-1} = 0$ and $M_0 = 1$ and using the difference equation $M_j = 5M_{j-1} - 2M_{j-2} - 8M_{j-3}$ repeatedly will give these moments. However, if a desk calculator is not available, it is probably better to write down each separate product as illustrated in the synthetic division process below.

	1	0	0	0	0	0	0	0	...		
-8				-8	-40	-184	-776	-3192	...		
-2			-2	-10	-46	-194	-798	...			
5		5	25	115	485	1995	8085	...			
	1	5	23	97	399	1617	6511	...	(Moments)		

Successive ratio approximations to 4 are approximately 5, 4.6, 4.22, 4.113, 4.053, 4.028, ...

To find the numerically smallest zero, one may use the difference equation to get the negative moments. Alternately, if $f(x)$ is a polynomial of degree n, negative moments of $f(x)$ are non-negative moments of $x^n f(1/x)$. In the present example,

$$x^3(x^{-3} - 5x^{-2} + 2x^{-1} + 8) = 8x^3 + 2x^2 - 5x + 1 \; .$$

However, such a change is not necessary, since expanding the reciprocal of $x^n f(1/x)$ in negative powers of x is closely related to expanding the reciprocal of $f(x)$ in positive powers of x.

In the present example, starting with M_{-3} the successive negative moments are:

$$-\frac{1}{8}, \frac{1}{32}, -\frac{11}{128}, \frac{29}{512}, -\frac{147}{2048}, \ldots$$

and the corresponding approximations to -1 are

$$-4, -\frac{4}{11}, -\frac{22}{29}, -\frac{116}{147}, \ldots \; .$$

One should note that an error in the computation of a particular moment will affect the value of all moments computed from that point on. However, (3) indicates that this merely amounts to having continued the computations with a new set of w_k's. Hence, such an error will not change the validity of (4).

If one knows that a certain zero of $f(x)$ is the one which is farthest from (nearest to) h, then the translation $x = h + y$ will make that zero correspond to the zero of the polynomial $f(h+y)$ whose absolute value is greatest (least). This procedure may be used to determine zeros of $f(x)$ other than those which are numerically greatest or least. Also if there is more than one zero which is numerically greatest (least), this procedure may be used to determine them.

Again, there is an alternate procedure to that of working directly with expansions of $c/f(x)$ in negative (positive) powers of y, where $y = x - h$. This procedure involves noting connections between the coefficients of these new expansions and those of the original expansions in powers of x and will lead to formulas which are well adapted to use with a calculator. The new coefficients are given by the following definition:

$$M_j(h) = \sum_{k=1}^{n} w_k(r_k - h)^j$$

is the jth moment about h of the zeros, r_k, of $f(x)$ with respect to the set of weights $\{w_k\}_{k=1,\cdots,n}$. $M_j(0) = M_j$. The following theorem gives the connection between the two sets of coefficients.

THEOREM 2: If $j \geqq 0$ then

$$M_j(h) = \sum_{k=0}^{j} \binom{j}{k}(-h)^k M_{j-k} \,.$$

If $j < 0$ and $h \neq r_i$ for $i = 1, \cdots, n$, then

$$M_j(h) = \sum_{k=0}^{m-1} \binom{k-j}{k} h^k M_{j-k} + h^m \sum_{i=1}^{n} w_i r_i^{j-m}(r_i - h)^j P_{-j,m}\left(\frac{h}{r_i}\right) \,.$$

Proof: If $j \geqq 0$,

$$M_j(h) = \sum_{i=1}^{n} w_i r_i^j \left(1 - \frac{h}{r_i}\right)^j = \sum_{i=1}^{n} w_i r_i^j \sum_{k=0}^{j} \binom{j}{k}\left(-\frac{h}{r_i}\right)^k = \sum_{k=0}^{j} \binom{j}{k}(-h)^k M_{j-k} \,.$$

The remainder of the theorem is proved with the aid of (1).

Example 2. Approximate $\sqrt{2}$, as a zero of $x^2 - 2$.

Method a: Let $x = y - 1$. Then $x^2 - 2 = y^2 - 2y - 1$ and the "non-negative" moments of this polynomial in y are 1, 2, 5, 12, 29, 70, $\cdots$. The first few approximations to $\sqrt{2}$ are

$$\frac{2}{1} - 1 = 1,\ \frac{5}{2} - 1 = 1.5,\ \frac{12}{5} - 1 = 1.4,\ 1\frac{5}{12},\ 1\frac{12}{29} \,.$$

Method b: For $x^2 - 2$, the values of M_j for $j = 0, 1, \cdots$ are 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, $\cdots$. Using theorem 2, one finds the corresponding $M_j(-1)$ to be 1, 1, 3, 7, 17, 41, 99, $\cdots$. An approximation for $\sqrt{2}$ is $(99/41) - 1$. If one had used 0, 1, 0, 2, 0, 4, 0, 8, $\cdots$ for the values of M_j, $j = 0, 1, \cdots$, then the corresponding values of $M_j(-1)$ would have been 0, 1, 2, 5, 12, $\cdots$.

Method c: The rate of convergence can be increased by magnifying the zeros since this will decrease $|r_i/r_1|$ for $i = 2, \cdots, n$. (See (4)). Let $x = (y - 15)/10$ and approximate the largest zero of $y^2 - 30y + 25$. Three approximations to this zero are 30/1, 875/30, 25500/875, and corresponding approximations to $\sqrt{2}$ are 1.5, 1.417, and 1.4142856.

Example 3. Approximate the zeros of $x^2 + x + 1$.

The moments keep repeating in value, as follows, 0, 1, -1, 0, 1, -1, 0, 1, −1, …. If one starts with $M_0 = 0$ and lets h equal $-i$, the moment set is 0, 1, $-1+2i$, $-3-3i$, $7-4i$, $4+15i$, -30, $15-56i$, …. An approximate value for one of the zeros is $((15-56i)/-30)-i$, and the zeros should be about $-.5 \pm .867i$.

It is probably worth noting that the Fibonacci sequence is the sequence of positive moments with respect to the zeros of x^2-x-1, provided that one takes $M_1 = M_2 = 1$.

The second part of theorem 2 is not very satisfactory for computational purposes. Hence, to find the zero of $f(x)$ which is nearest h, it is better to work directly with the negative moments of the polynomial $f(y+h)$. However, one can use the first part of theorem 2 to locate the zero of $f(x)$ whose reciprocal is farthest from h, since negative moments with respect to a set of zeros of $f(x)$ are positive moments with respect to their reciprocals.

Translations may be used, as already indicated, to get approximations to zeros of polynomials other than those of greatest or least absolute value or to approximate zeros of greatest (least) absolute value when there is more than one such zero. However, a method of making these approximations which may prove to be simpler, computationally, will now be developed. This method makes use of an extension of (4).

$$M_j = \sum_{i=1}^{m} w_i r_i^j + w_m r_m^j \sum_{i=m+1}^{n} \frac{w_i}{w_m}\left(\frac{r_i}{r_m}\right)^j .$$

Hence, if the zeros of $f(x)$ are so ordered that $|r_i| \underset{(\leqq)}{\geqq} |r_k|$ for $i<k$ and $|r_{m+1}| \underset{(<)}{>} |r_m|$, then $\sum_{i=1}^{m} w_i r_i^j$ will approximate M_j as $j \to \underset{(-\infty)}{\infty}$. Consequently, the m numerically $\underset{\text{(smallest)}}{\text{largest}}$ zeros of $f(x)$ will be approximated by the zeros of the polynomial

$$\begin{vmatrix} 1 & x & \cdots & x^{m-1} & x^m \\ M_{j-m} & M_{j-m+1} & \cdots & M_j & M_{j+1} \\ \vdots & \vdots & & \vdots & \vdots \\ M_j & M_{j+1} & \cdots & M_{j+m} & M_{j+m+1} \end{vmatrix} \quad \text{as } j \to \underset{(-\infty)}{\infty} .$$

Let $D_{m,j}$ be the coefficient of x^m in this new polynomial. Then,

$$\frac{D_{m,j+1}}{D_{m,j}}$$

approximates the product of these zeros.

Example 4. Approximate all the zeros of x^3-5x^2+2x+8.

This is a continuation of example 1. If one writes the positive moments in a row and then writes them again, moving each moment one position to the left, the $D_{2,j}$ are readily computed from two adjacent columns.

M_{j-1}	1	5	23	97	399	1617	6511	26129	
M_j	5	23	97	399	1617	6511	26129	104687	
$D_{2,j}$		−2	−44	−232	−2352	−16899	−142528	−1107584	

$$\frac{-1107584}{-142528} \cdot \frac{26129}{104687} \approx 1.94 \; .$$

The true value of this zero is 2.

Generally,

$$(5) \qquad r_m \approx \frac{D_{m,j+1} D_{m-1,j}}{D_{m,j} D_{m-1,j+1}}, \quad \text{if } |r_i| \underset{(\geqq)}{\leqq} |r_j| \text{ for } i < j \text{ and } |r_i| \neq |r_m| \text{ for } i \neq m \, .$$

Thus far, polynomials with repeated factors have not been considered. If $f(x)$ has the factor $x - r_k$ repeated n_k times then the partial fraction expansion of $g(x)/f(x)$ will involve terms of the type $c_{k,j}/(x-r_k)^j$ for $j = 1, \cdots, n_k$ and if $i \geqq n_k$, then the coefficient of x^{-i-1} in the expansion corresponding to (1) will consist of terms of the form

$$\sum_{j=1}^{n_k} c_{k,j} \binom{i}{i+1-j} r_k^{i+1-j} = \sum_{j=1}^{n_k} c_{k,j} \binom{i}{j-1} r_k^{1-j} r_k^i \, .$$

M_i will consist of a sum of such terms. Since the ith term of (1) approaches zero as i becomes infinite if $|u| < 1$, it follows that the ratio approximation can still be used under the same conditions as before, although the set of M_i's is no longer the ith moment with respect to a fixed set of weights.

$$\frac{M_{i+1}}{M_i} \approx \frac{r_1 \sum_{j=1}^{n_1} c_{1,j} \binom{i+1}{j-1} r_1^{1-j}}{\sum_{j=1}^{n_1} c_{1,j} \binom{i}{j-1} r_1^{1-j}} \approx r_1 \, .$$

Similarly, the development in the paragraph before example 4 is valid and translations are still permissible.

Example 5. Approximate the largest zero of $x^3 - 3x^2 + 4$.

Starting with three moments of 0, the following moments are 1, 3, 9, 23, 57, 135, 313, 711, 1593. (1593/711) ≈ 2.24. A suitable translation should improve the rapidity of convergence. Also, one might round off the last three moments to 3, 7, 17, and start once more, getting the new moments: 36, 80, 176, 384, 832 to get the approximation 2.17.

In summary, to approximate the zero of a polynomial which is farthest from (nearest to) h, expand the reciprocal of that polynomial in negative (positive) powers of $x-h$ and use the quotient of the coefficients of two consecutive powers of $x-h$, the coefficient of the greater power being in the denominator, as an approximation. Synthetic division may be used to determine the coefficients. This is the viewpoint which is probably most adaptable to an elementary algebra course. Note that one could then present this as a possible method of developing tables of roots, exponential tables ($10^{.3}$ is a solution of $x^{10}-1000=0$, and one might find the zero nearest 2) and trigonometric tables (with the aid of De Moivre's theorem).

Finally, the alternate viewpoint has been that of using a particular solution of a linear homogeneous difference equation with constant coefficients to approximate the zeros of its auxiliary polynomial. This is the reverse procedure to that usually found in books on finite differences and numerical analysis. These approximate solutions to the zeros of the polynomial could now be used to get approximations to the general solution of the difference equation. Both methods have generated moments, a procedure used in statistics.

Note: A. Ostrowski deals with this method of approximating zeros of a polynomial from the complex variables viewpoint in Appendix J of *Solutions of Equations and Systems of Equations,* Academic Press, 1960.

Answers *to Quickies* *on page 193.*

A 297. The lines are equal chords tangent to a circle concentric with the given circle. If more than two lines passed through one point, there would be more than two tangents from an external point to a circle, which is impossible.

A 298. We have $(1!)^2 = 1^1$ and $(2!)^2 = 2^2$, but for $n > 2$ we have

$$\frac{(n!)^2}{n^n} = \frac{n\cdot 1}{n}\cdot\frac{(n-1)2}{n}\cdot\frac{(n-2)3}{n}\cdots\frac{2(n-1)}{n}\cdot\frac{1\cdot n}{n} > 1$$

since all of the fractions in the product are $\geqq 1$, and at least one fraction exceeds 1. Therefore $(n!)^2 > n^n$ for $n > 2$.

TEACHING OF MATHEMATICS

Edited by ROTHWELL STEPHENS, Knox College

This department is devoted to the teaching of mathematics. Thus, articles of methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Rothwell Stephens, Mathematics Department, Knox College, Galesburg, Illinois.*

A NOTE ON A THEOREM IN COMPLEX VARIABLES AND APPLICATIONS

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Consider the complex polynomials $P(z)$ and $Q(z)$ where

$$P(z) = \sum_{K=0}^{K=n} a_K z^{n-K}, \quad a_0 = 1,$$

and

$$Q(z) = \sum_{r=0}^{r=m} b_r z^{m-r}, \quad b_0 = 1.$$

Let C be a closed curve which encloses all the zeroes of $Q(z)$, then

$$\oint_C \frac{P(z)}{Q(z)}\,dz = 0 \quad \text{if } m-n \geqq 2$$
$$= 2\pi i \quad \text{if } m-n = 1 .$$

The proof of this follows directly from a theorem and a "rule" by Kaplan: [1]

"Theorem 46. If $f(z)$ is analytic in the extended z plane except for a finite number of singularities, then the sum of all residues of $f(z)$ (including ∞) is zero."

"Rule V. If $f(z)$ has a zero of first order at ∞, then

$$\operatorname{Res}[f(z), \infty] = -\lim_{z\to\infty} zf(z)$$

If $f(z)$ has a zero of second or higher order at ∞, the residue at ∞ is 0."

In view of this theorem, many problem exercises given in contemporary mathematics texts can be solved by inspection, e. g., the answers to all of the following problems can be seen to be equal to zero:

I. $$\oint \frac{1}{z^4+1}\,dz, \quad \text{on path } |z| = 2 .$$ (Reference 2)

II. $$\oint \frac{1}{z^2(z+1)}\,dz, \quad \text{on path } |z| = 2 .$$ (Reference 2)

III. $\displaystyle\int \frac{z+1}{z^3-2z^2}\,dz$, around circle $|z-2-i| = 2$. (Reference 3)

IV. $\displaystyle\int \frac{dz}{z(z+4)}$, on path $|z+2| = 3$. (Reference 4)

Furthermore, the answer to the following can be seen to be $2\pi i$ by inspection:

V. $\displaystyle\oint \frac{z^7}{(z^4+1)^2}\,dz$, $|z| = 2$. (Reference 2)

In addition to determining the value of certain integrals by inspection, utilization of the theorem simplifies the calculation of certain other integrals, e. g., the computation of the integral

$$\oint_{|z|=2} \frac{1}{(z-1)^3(z-7)}\,dz \qquad \text{(Reference 5)}$$

is easily accomplished by noting that for a path outside the poles, the integral is zero. Consequently, the desired integral is the negative of the integral about $z = 7$ which gives the value to be

$$-2\pi i\left\{\frac{1}{(z-1)^3}\right\}\Big|_{z=7} = -\frac{\pi i}{108}\,.$$

An interesting and useful corollary to this theorem is the following:

If the function, $P(z)/Q(z)$, has only simple poles, then the sum of coefficients of the partial fraction expansion of the function will equal zero if $m-n \geqq 2$ and will equal one if $m-n = 1$.

The utility of this in checking the correctness of many partial fraction expansions is evident.

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NOTE ON $\int_a^x t^y \, dt$

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The marked difference in the formula for

$$\int_a^x t^y \, dt \qquad 0 < a < x$$

when $y = -1$ as compared with the case when $y \neq -1$ seems to imply that there is a "discontinuity." That this is not the case may be shown as follows:

Let

$$f(y) = \int_a^x t^y \, dt \qquad 0 < a < x$$

so that

$$f(y) = \frac{x^{y+1} - a^{y+1}}{y+1} \qquad \text{for } y \neq -1$$

and applying L'Hospital's Rule we find:

$$\lim_{y \to -1} f(y) = \lim_{y \to -1} \frac{(\ln x)x^{y+1} - (\ln a)a^{y+1}}{1}$$

$$= \ln x - \ln a \, .$$

There are various ways given in the textbooks on calculus to arrive at

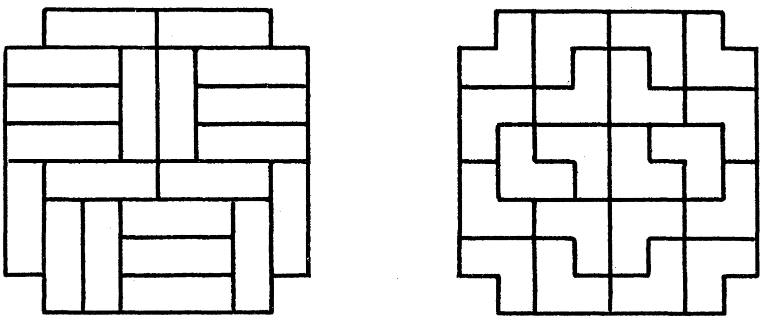
$$f(-1) = \int_a^x t^{-1} \, dt = \ln x - \ln a \, .$$

Hence $f(y)$ is indeed continuous at $y = -1$.

TWO TROMINO TESSELLATIONS

CHARLES W. TRIGG, Los Angeles City College

Since 3 does not divide 64 or 62 without remainder, a complete checkerboard or one with two opposite corner squares removed cannot be covered by trominoes, i. e. pieces composed of three connected squares. A checkerboard with a square removed from each corner can be covered, and in a number of ways. One arrangement with straight trominoes and one with right trominoes are shown. The first arrangement contains two fault lines and the second, one fault line. A fault line is one along which a portion of the figure may be slid without otherwise disturbing the relative arrangement of the pieces.



For a general discussion of such tessellations see:

S. W. Golomb, *Checkerboards and Polyominoes*, American Mathematical Monthly, Vol. 61, December, 1954, pp. 675-82.
S. W. Golomb, *The General Theory of Polyominoes,* Recreational Mathematics Magazine, No. 4, August, 1961, pp. 3-12; No. 5, October, 1961, pp. 3-12; No. 6, December, 1961, pp. 3-22.

MISCELLANEOUS NOTES

Edited by ROY DUBISCH, University of Washington

Articles intended for this department should be sent to *Roy Dubisch, Department of Mathematics, University of Washington, Seattle, Washington.*

NUMBER BASES AND BINOMIAL COEFFICIENTS

JOHN M. HOWELL and ROBERT E. HORTON, Los Angeles City College

The division algorithm for changing numbers from their representation with base 10 to representations with other bases is quite satisfactory for most purposes. This note is presented to show an alternate procedure for changing numbers written in the decimal system to numbers with bases of 9, 3, or 2, making use of binomial coefficients. The interest lies in the way this procedure reveals a relationship between the number base and the binomial coefficients.

The procedure develops from the fact that the integral powers of 10 can be expressed in the following fashion:

$$10^n = (9+1)^n = (3^2+1)^n = 2^n(2^2+1)^n .$$

Applying the binomial expansion in each of these cases gives

(1) $$10^n = \sum_{x=0}^{n} \binom{n}{x} 9^x$$

(2) $$10^n = \sum_{x=0}^{n} \binom{n}{x} 3^{2x}$$

(3) $$10^n = 2^n \sum_{x=0}^{n} \binom{n}{x} 2^{2x} .$$

Consider equation (1) with $n = 3$, say. This is

$$10^3 = (9+1)^3 = (9)^3 + 3(9)^2 + 3(9) + 1 .$$

It is evident that the binomial coefficients can be used to represent the digits in the number written with base 9 that is equivalent to 10^3. In this way we can build a table of integral powers of 10 written in base 9.

Note that when the binomial coefficient exceeds 8, the nonary number must be formed by writing this coefficient in base 9 and carrying to the left where necessary. See Table I.

Table I

Decimal	Intermediate Step	Nonary
10^0	1	1
10^1	1 1	1 1
10^2	1 2 1	1 2 1
10^3	1 3 3 1	1 3 3 1
10^4	1 4 6 4 1	1 4 6 4 1
10^5	1 5 (10) (10) 5 1	1 6 2 1 5 1
10^6	1 6 (15) (20) (15) 6 1	1 7 8 3 6 6 1

Next consider equation (2) with $n = 3$.

$$10^3 = (3^2+1)^3 = (3^2)^3 + 3(3^2)^2 + 3(3^2) + 1$$
$$= (3)^6 + 0 + 3(3)^4 + 0 + 3(3)^2 + 0 + 1 .$$

It is evident that such an expansion will lead to a table looking like the Pascal triangle with zeros inserted between consecutive entries.

Table II

Decimal	Intermediate Step	Ternary
10^0	1	1
10^1	1 0 1	1 0 1
10^2	1 0 2 0 1	1 0 2 0 1
10^3	1 0 3 0 3 0 1	1 1 0 1 0 0 1
10^4	1 0 4 0 6 0 4 0 1	1 1 1 2 0 1 1 0 1
10^5	1 0 5 0 (10) 0 (10) 0 5 0 1	1 2 0 0 2 0 1 1 2 0 1
10^6	1 0 6 0 (15) 0 (20) 0 (15) 0 6 0 1	1 2 1 2 2 1 0 2 0 2 0 0 1

To get the ternary expression from the intermediate step it is necessary to change all entries in this quasi-Pascal triangle greater than 2 into their ternary equivalents and carry.

Finally, the binary procedure using equation (3) leads to Table III.

Table III

Decimal	Intermediate Step	Binary
10^0	1	1
10^1	1 0 1 0	1010
10^2	1 0 2 0 1 0 0	1100100
10^3	1 0 3 0 3 0 1 0 0 0	1111101000
10^4	1 0 4 0 6 0 4 0 1 0 0 0 0	10011100010000
10^5	1 0 5 0 (10) 0 (10) 0 5 0 1 0 0 0 0 0	11000011010100000

The following will illustrate the application of the procedure.

Problem: Change 293 to the bases 9, 3, and 2. For base 9 write,

$$\begin{aligned} 293 &= 200 + 90 + 3 \\ &= 2(10^2) + 9(10) + 3(1) \\ &= 2(121) + 10(11) + 3(1) \quad \text{(Base 9)} \\ &= 242 + 110 + 3 \quad \text{(Base 9)} \\ &= 355 \quad \text{(In base 9.)} \end{aligned}$$

Note that in the third line it was necessary to change the 9 of the preceding line into its nonary equivalent.

For base 3, write

$$\begin{aligned} 293 &= 2(10^2) + 9(10) + 3(1) \\ &= 2(10201) + 100(101) + 10(1) \quad \text{(Base 3)} \\ &= 30512 \quad \text{(Base 3)} \\ &= 101212 \quad \text{(In base 3.)} \end{aligned}$$

For base 2, write

$$\begin{aligned} 293 &= 2(10^2) + 9(10) + 3(1) \\ &= 10(1100100) + 1001(1010) + 11(1) \quad \text{(Base 2)} \\ &= 11001000 + 1011010 + 11 \quad \text{(Base 2)} \\ &= 100100101 \quad \text{(In base 2.)} \end{aligned}$$

It is true that $10^n \equiv (a+b)^n$ could be expanded by the binomial expansion for any two integers a and b which satisfy the identity. However, the expansion is not particularly interesting if $b \neq 1$.

COMMENTS ON PAPERS AND BOOKS

Edited by HOLBROOK M. MACNEILLE, Case Institute of Technology

This department will present comments on papers published in the MATHEMATICS MAGAZINE, lists of new books, and reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent to *Holbrook M. MacNeille, Department of Mathematics, Case Institute of Technology, Cleveland 6, Ohio.*

BOOK REVIEWS

Non-Euclidean Geometry. By Stefan Kulczycki. Pergamon Press, New York, 1961, 208 pp., $10.

The author restricts himself to hyperbolic geometry which he also calls Lobatchevskian geometry. In his historical introduction (Chapter I) he selects material to stimulate and prepare the reader for the later developments. If we were to disregard the historical background we could say that the introductory chapter deals with a number of assumptions equivalent to the Fifth Postulate (or axiom of Euclid). Here, and again in Chapter III, we find discussions of the logical structure of geometry versus empirical knowledge of space.

In Chapter II it becomes clear why most of the axioms are not stated explicitly. The author relies on the reader's previous experience with "absolute" geometry and encourages him to look further into the matter. Thus, it is the absolute theorems that form the foundation of the logical structure, which may well be a commendable approach for so short an account of non-Euclidean geometry. Among other things the second chapter contains Hjelmslev's theorem and an interesting mapping of the hyperbolic plane into a hyperbolic circle. This mapping, also extended to three dimensions, proves to be most fruitful in the remainder of the book. Basic properties of parallels and non-intersectors, for example, follow easily. The author also discusses the following topics: angle of parallelism, area of a polygon, regular polygons, equidistant curves, limiting curves or horocycles, horosphere. The chapter closes on this high note: the geometry on the horosphere is Euclidean.

The first part of Chapter III is devoted to Lobatchevsky's method of deriving hyperbolic trigonometry which to this day seems powerful and efficient. All it takes is the formula for the arcs of concentric horocycles and the relation between a segment and its angle of parallelism, the latter being found with the aid of the horosphere. Trigonometry is then used for the Lambert quadrangle (three right angles), problems on regular polygons, equidistant curves. Also included is a complete derivation of spherical trigonometry in hyperbolic space and some analytic geometry. As a finishing touch a model provides the evidence for the consistency of hyperbolic geometry.

There are no exercises in the book. A few misprints will do little harm. The reader is being guided throughout the book and motivated to continue. The author's expository skill makes his approach to a classical subject appear fresh and newer than it is. He surely is inspired by what is, after all, one of the great ideas of man.

Curtis M. Fulton, University of California, Davis

Numbers: Rational and Irrational. By Ivan Niven. Random House, New York, 1961, viii + 136 pages, paper back, $1.95.

What is Calculus About? By W. W. Sawyer. Random House, New York, 1961, viii + 118 pages, paper back, $1.95.

An Introduction to Inequalities. By Edwin Beckenbach and Richard Bellman. Random House, New York, 1961, viii + 133 pages, paper back, $1.95.

These are the first three volumes of the New Mathematical Library sponsored by the School Mathematics Study Group. The books in this series are intended to make some important mathematical ideas interesting and understandable to high school students and laymen.

Niven's book is admirably suited for this audience, and may be read with profit by persons with more extensive mathematical training. The first five chapters are concerned with natural numbers and integers, rational numbers, real numbers, irrational numbers, trigonometric and logarithmic numbers. The treatment is inductive and intuitive and presents little difficulty to the reader. The final two chapters on the approximation of irrationals by rationals and the existence of transcendental numbers are more advanced. But the theorems are understandable, even if a youthful reader may not be able to follow all the detail of their proofs. Appendices contain proofs of the existence of infinitely many primes, the unique factorization theorem, and the existence of transcendental numbers. The reader will benefit by the well chosen problems (with answers and suggestions), the comments on the nature of proof, and the careful treatment of an incommensurable case in geometry. Various historical comments, such as those on the theorem proved by Roth in 1955, serve to show the current interest in irrational numbers and their approximation by rationals.

Sawyer's book is intended for the high school student unacquainted with the ideas of calculus. Many college students after having had a course in calculus would profit by reading this book and thereby re-examining calculus from an intuitive point of view. Sawyer bases the idea of a derivative upon the more familiar ideas of speed and of slope of a curve. Second derivatives are introduced thru acceleration and curvature. A simple maximum problem is solved. Finally the "reverse problem" of integration or anti-differentiation is briefly discussed and the calculation of areas and volumes is mentioned. In the final chapter, "Intuition and Logic" and in an appendix, "Guide to Further Study", the author emphasizes that he has given the reader merely a brief intuitive survey of calculus. In order for him to master the subject and use its tools correctly, he will be required to do considerably more reading and studying.

The volume by Beckenbach and Bellman will probably prove to be a more difficult book for high school students and laymen to read and understand. The first three chapters emphasize the axiomatic basis of inequalities. Theorems on inequalities, on the properties of the real number system, and on absolute value are proved. The fourth chapter uses these theorems to derive many of the classical inequalities of analysis. In chapter five various problems in maximization and minimization are solved by using inequalities rather than by the more conventional methods of calculus. Inequalities are even used to obtain the equations of tangents to an ellipse! The final chapter is devoted to generalizations of the concept of euclidean distance. Here the triangle inequality is of interest.

The reviewer feels that SMSG is to be praised for making possible the publication of this series. Although written for an audience with little formal training in mathematics, these books should serve to recall to others that advanced problems may often be solved by elementary methods and that the elementary part of mathematics has its own fascinating aspects.

Harry M. Gehman, University of Buffalo

Tables of All Primitive Roots of Odd Primes Less Than 1000. By Roger Osborn. University of Texas Press, Austin, 1961, 70 pages, hard cover, $3.00.

These tables of primitive roots were computed on an IBM 650 computer and printed on an IBM 407 directly from the punched cards. Enough varied validity checks were made to insure that the tables are virtually error-free. The type is large and clear so that the well-arranged tables are easily read. Previous tables extended only to $p = 353$, so that these carefully computed tables to $p = 1000$ should be welcomed by investigators in number theory. Current and past studies can be extended with the aid of these tables and other projects may well be initiated.

Charles W. Trigg, Los Angeles City College

BOOKS RECEIVED FOR REVIEW

A Book of Curves. By E. H. Lockwood. Cambridge University Press, New York, 1961, xi + 199 pages, $4.95.

Arithmetic in Maya. By George I. Sánchez, 2201 Scenic Drive, Austin 3, Texas, 1961, 74 pages, $5.00.

Mathematical Snapshots. By Hugo Steinhaus. Oxford University Press, New York, 1960, 328 pages, $6.75.

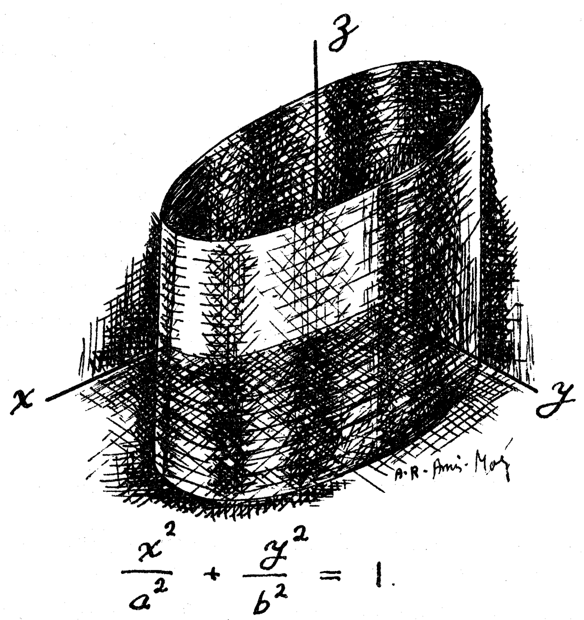
Introduction to Geometry. By H. S. M. Coxeter. John Wiley and Sons, Inc., New York, 1961, xiv + 443 pages, $9.95.

Cartesian Geometry of The Plane. By E. M. Hartley. Cambridge University Press, 1960, xi + 135 pages, \$5.50.

Analytical Quadrics. By Barry Spain. Pergamon Press, New York, 1960, ix + 135 pages, \$5.50.

Introduction to Analytic Geometry and Linear Algebra. By Arno Jaeger. Henry Holt and Company, 1960, xii + 305 pages, \$6.00.

Mathematical Discovery. Volume I. By George Polya. John Wiley and Sons, Inc., New York, 1962, xvi + 216 pages, hard cover, \$4.75.



PROBLEMS AND SOLUTIONS

Edited by ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.*

PROPOSALS

481. *Proposed by Daniel I. A. Cohen, Central High School, Philadelphia, Pennsylvania.*

Prove that the r-th m-gonal number can never be the m-th r-gonal number unless $r = m$.

482. *Proposed by Brother U. Alfred, St. Mary's College, California.*

Given a rectangular array of equally spaced points with $n+1$ points along one side and $m+1$ points along the other. If $m = \Pi p_i^{\alpha_i}$ in terms of prime factors and $n = \Pi q_j^{\beta_j}$ in terms of prime factors, determine an expression for the number of lines passing through at least three points of the array, one point being a corner and a second one of the other points on the periphery.

483. *Proposed by Dermott A. Breault, Sylvania Electric Products, Waltham, Massachusetts.*

Given that $w = f(z)$ is conformal in a region R, and $z = f^{-1}(w)$ is conformal in R'. Set $w = \phi(x, y) + i\psi(x, y)$ and $z = x(\phi, \psi) + iy(\phi, \psi)$. Show that the Jacobians $\dfrac{\partial(\phi, \psi)}{\partial(x, y)}$ and $\dfrac{\partial(x, y)}{\partial(\phi, \psi)}$ are not zero in R and R' respectively.

484. *Proposed by Leon Bankoff, Los Angeles, California.*

A square $ADEB$ is constructed externally on the hypotenuse AB of a right triangle ABC, $(CB > CA)$. CE cuts AB at S, and a perpendicular to AB at S cuts CB at Q. T is the foot of the bisector of angle BCA, and P is the foot of the perpendicular from T on CB. TP cuts QS at R. Show that $QR = TS$.

485. *Proposed by R. P. Steinkirk, University of Missouri.*

$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1+(1/p)}$ converges for all p greater than zero. What about $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1+(1/n)}$?

486. *Proposed by M. Rumney, London, England.*

Given three different positive integers N_1, N_2, and N_3. Find a partition of N_1 into three different positive integers a_{11}, a_{12}, a_{13}, N_2 into three different positive integers a_{21}, a_{22}, a_{23}, and N_3 into three different positive integers a_{31}, a_{32}, a_{33} so that the determinant $|a_{ij}| = 0$, $i = 1, 2, 3$ and $j = 1, 2, 3$.

487. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Find the square root of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Errata.

Problem 470, Vol. 35, No. 1, January 1962, p. 55 should read: Prove that

$$\sum_{n=0}^{N} (-1)^n \frac{(1+2N-n)!\,(1+N)!}{(1+n)!\,(N-n)!\,(1+N-n)!\,(2+2N)!} = \frac{1}{(N+2)!} .$$

The comments on Problems 428 and 432 made in Vol. 35, No. 1, January 1962 were made by C. F. Pinzka and D. Moody Bailey, respectively.

SOLUTIONS

Late Solutions

445, 446, 447, 448, 453. *Josef Andersson, Vaxholm, Sweden.*

457. *Josef Andersson, Vaxholm, Sweden; and W. C. Waterhouse, Harvard University.*

454, 455, 458. *W. C. Waterhouse, Harvard University.*

459. *Josef Andersson, Vaxholm, Sweden; Gilbert Labelle, Collège de Longueuil, Canada; and W. C. Waterhouse, Harvard University.*

A Product of Complex Roots

460. [November 1961] *Proposed by Robert P. Goldberg, Brooklyn, New York.*

Given a regular polynon $A_1A_2 \cdots A_n$ inscribed in a unit circle. Prove that

$$\prod_{i=2}^{n} A_1A_i = n .$$

Solution by F. D. Parker, University of Alaska.

Without loss of generality, we can locate the vertices at the complex roots of $z^n = 1$, such that A_1 is on the positive real axis. Then

$$\prod_{i=2}^{n} A_1A_i = \left|\prod_{i=2}^{n} (z_i - 1)\right| = \prod_{i=2}^{n} |(z_i - 1)| .$$

Expanding this and using the well-known relations between the roots of an equation and its coefficients, the result follows quickly.

Also solved by Josef Andersson, Vaxholm, Sweden; Joseph Bohac, St. Louis, Missouri; Maurice Brisebois, University of Sherbrooke, Canada; Leonard Carlitz, Duke University; E. E. Morrison, University of Cincinnati; C. F. Pinzka, University of Cincinnati; Roger D. H. Jones, College of William and Mary; David L. Silverman, Beverly Hills, California; and the proposer.

Chemical Cryptarythm

461. [November 1961] *Proposed by Maxey Brooke, Sweeny, Texas.*
Prove the cryptarythm $[NA]\cdot[CL] \neq SALT$ in the base 6.

Solution by J. A. H. Hunter, Toronto, Canada.
If there is a solution for the alphametic $(NA)\cdot(CL) = SALT$ in base 6, then $N \neq A \neq C \neq L \neq S \neq$ zero, so $T =$ zero. Hence, using normal base-10 numerals, $(A, L) = (2, 3)$ or $(3, 4)$ interchangeably for either part of values. Also, $S < N$, $S < C$, so $S = 1$.
If $(A, L) = (3, 4)$, with $(N, C) = (2, 5)$ interchangeably, we would have $S = 2$ which is impossible.
Hence there is no solution.

Also solved by Daniel I. A. Cohen, Philadelphia, Pennsylvania; Harry M. Gehman, University of Buffalo; Dee Fuller, Davidson, North Carolina; Gilbert Labelle, Collège De Longueuil, Canada; C. C. Oursler, Southern Illinois University; C. F. Pinzka, University of Cincinnati; David L. Silverman, Beverly Hills, California; Paul Stygar, Yale University; Charles W. Trigg, Los Angeles City College; and the proposer.
Pinzka noted that neither $NA\cdot CN = RIP$ nor $K\cdot CN = PDQ$ has a solution in base 6.

Greatest of Three

462. [November 1961] *Proposed by Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.*
It is well known that $\max(x, y) = \frac{1}{2}\{|x-y| + x + y\}$. Find a similar expression for $\max(x, y, z)$.

Solution by J. L. Brown, Jr., Pennsylvania State University.
Since $\max(x, y, z) = \max[z, \max(x, y)]$, application of the given formula yields

$$\max(x, y, z) = \tfrac{1}{2}\left\{\left|z - \frac{|x-y|+x+y}{2}\right| + z + \frac{|x-y|+x+y}{2}\right\}$$
$$= \tfrac{1}{4}\{|2z - |x-y| - x - y| + 2z + |x-y| + x + y\}.$$

Also solved by Homer F. Bechtell, Lebanon Valley College, *Pennsylvania; L. Carlitz, Duke University; David Chale, University of California, Berkeley; Daniel I. A. Cohen, Philadelphia, Pennsylvania; F. D. Parker, University of Alaska; C. F. Pinzka, University of Cincinnati; Lawrence A. Ringberg, Eastern Illinois State University; David L. Silverman, Beverly Hills, California; and the proposer.* One incorrect solution was received.

Comment by David Chale, University of California, Berkeley. Rewriting $\max(x, y, z)$ as $\max[\max(x, y), z]$ we see that $\max(x_1, x_2, x_3, x_4) = \max\{\max[\max(x_1, x_2), x_3], x_4\}$ and in general

$$\max(x_1, x_2, \cdots, x_n) = \max[\max(\max\cdots x_{n-1}), x_n] .$$

Thus $\max(x_1, x_2, \cdots, x_n)$ is expressible in a form similar to $\max(x, y, z)$, though with a complexity of absolute value signs.

It is easy to see that $\min(x, y, z) = -\max(-x, -y, -z)$ and in general $\min(x_1, x_2, \cdots, x_n) = -\max(-x_1, -x_2, \cdots, -x_n)$. Next let $\operatorname{int}(x, y, z)$ denote the second largest of the three numbers. Then it is evident that

$$\operatorname{int}(x, y, z) = -[\max(x, y, z) + \min(x, y, z) - x - y - z] .$$

Note that

$$\operatorname{int}(x, y, z) = [x + y + z - \max(x, y, z) - \min(x, y, z)]$$
$$= [x + y + z - \max(x, y, z) + \max(-x, -y, -z]$$

therefore

$$\operatorname{int}(x, y, z) = \tfrac{1}{4}\{2x + 2y + ||x-y| - x - y + 2z| - ||x-y| + x + y - 2z|\} .$$

A different approach is required to obtain the kth largest number of $(x_1, x_2, \cdots, x_n)$. Define $f_n^k(x_1, x_2, \cdots, x_n)$ as the kth largest of $(x_1, x_2, \cdots, x_n)$. Note that $f_3^2(x_1, x_2, x_3) = \operatorname{int}(x_1, x_2, x_3)$.

Theorem : For $(x_1, x_2, \cdots, x_n)$ all distinct real numbers, $n-1 \geqq k \geqq 2$,

$$f_n^k(x_1, x_2, \cdots, x_n) = f_3^2[f_{n-1}^{k-1}(x_1, x_2, \cdots, x_3), f_{n-1}^k(x_1, x_2, \cdots, x_n), x_n] .$$

Proof:

$$f_n^k = f_{n-1}^k \quad \text{if} \quad f_{n-1}^k > x_n ,$$
$$f_n^k = f_{n-1}^{k-1} \quad \text{if} \quad f_{n-1}^{k-1} < x_n ,$$
$$f_n^k = x_n \quad \text{if} \quad f_{n-1}^{k-1} > x_n > f_{n-1}^k .$$

Completing the inequalities :

$$f_n^k = f_{n-1}^k \quad \text{if} \quad f_{n-1}^{k-1} > f_{n-1}^k > x_n ,$$
$$f_n^k = f_{n-1}^{k-1} \quad \text{if} \quad x_n > f_{n-1}^{k-1} > f_{n-1}^k ,$$
$$f_n^k = x_n \quad \text{if} \quad f_{n-1}^{k-1} > x_n > f_{n-1}^k .$$

On inspection of the latter inequalities it is seen that

$$f_n^k = f_3^2[f_{n-1}^k, f_{n-1}^k, x_n] \,.$$

The Deadbeats

463. [November 1961] *Proposed by D. L. Silverman, Fort Meade, Maryland.*

A debt exists between each pair of members of a certain club, though no member is indebt to all of the others. A member is considered a "deadbeat" if every other member is a creditor or a creditor of a creditor of his. Show that the club has at least three deadbeats.

Solution by John W. Moon, University of Alberta, Canada.

Denote the members of the club by $P_1, P_2, \cdots, P_n$, $n \geq 3$, and let the score of any member be the number of other members who are indebted to him. It is known (see E. G. Kemeny, Snell, and Thompson, *An Introduction to Finite Mathematics*, p. 317) that if P_1, say, is the lowest scored member then every other member is either a creditor, or a creditor of a creditor of P_1. Essentially the same argument can be used to show that P_2 and P_3 have this same property, where P_2 is the lowest scored member indebted to P_1 and P_3 is the lowest scored member indebted to P_2. The existence of these members is assured by the hypothesis that no member had a score of zero. As P_1, P_2, and P_3 are clearly distinct members the presence of at least three "deadbeats" in every such club is implied.

Without additional assumptions the above argument cannot be reached again with respect to P_3 since there is no assurance that the new "deadbeats" are different from those already counted. That three is a best possible result, in a sense, is shown by the club in which P_i is a creditor of P_j if, and only if, $i > j$, except that P_1 is a creditor of P_3.

Leo Moser (unpublished manuscript) has shown that as $n \to \infty$ the probability that a round-robin tournament on n points has diameter two tends to one. In the terminology of the present problem this implies, under the assumption that P_i is equally as likely to be indebted to P_j, $i \neq j$, as vice versa, that as $n \to \infty$ the probability that every member of the club is a "deadbeat" tends to one.

Also solved by the proposer.

Extended Pythagorean Numbers

464. [November 1961] *Proposed by D. Rameshwar Rao, Secunderabad, Andhra Pradesh, India.*

If $a^2+b^2 = c^2$, prove that one can find integers $d, e, f, g, \cdots$ and $d_1, e_1, f_1, g_1, \cdots$ such that $a^2+b^2+d^2 = d_1^2$, $a^2+b^2+d^2+e^2 = e_1^2$, $a^2+b^2+d^2+e^2+f^2 = f_1^2$ and so on, where $d, d_1; e, e_1; f, f_1; \cdots$ are consecutive integers.

Solution by L. Carlitz, Duke University.

It is necessary to add to the hypothesis that c is odd.

Given $a^2+b^2=c^2$, with c odd, the equation

$$a^2+b^2+d^2=(d+1)^2$$

is equivalent to $a^2+b^2=2d+1$. Since c is odd this equation uniquely determines d. Moreover since $a^2+b^2 \equiv 1 \pmod 4$, d is even.

Next the equation

$$a^2+b^2+d^2+e^2=(e+1)^2$$

is equivalent to

$$a^2+b^2+d^2=2e+1 .$$

Since $a^2+b^2+d^2 \equiv 1 \pmod 4$, e is uniquely determined and is even.

This process can evidently be continued as far as desired.

Also solved by Brother U. Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; William Bart, Loyola University, Chicago; Dee Fuller, Davidson, North Carolina; Herbert R. Leifer, Pittsburgh, Pennsylvania; C. C. Oursler, Southern Illinois University; Paul Schotten, Calvin College, Michigan; David L. Silverman, Beverly Hills, California; Charles W. Trigg, Los Angeles City College; W. C. Waterhouse, Harvard University; Paul Stygar, Yale University; and the proposer.

Digit Square Sums

466. [November 1961] *Proposed by E. P. Starke, Rutgers University.*

Let A be a positive integer of r digits given by

$$A=\sum_{i=1}^{r} x_i 10^{i-1}$$

and define

$$D(A)=\sum_{i=1}^{r} x_i^2 .$$

By $D^n(A)$, where n is a positive integer, we mean the result of applying the operator, D, n successive times to A. Prove that for every A there exists an n such that $D^n(A)=1$ or 4.

(Cf. E 718 *American Mathematical Monthly.*)

Solution by Brother U. Alfred, St. Mary's College, California.

On the first operation, the number will be reduced to a quantity less than $100r$. For example, a number having a million digits would have $D(A)$ less than 100 million. Very quickly the value resulting from successive operations will be less than 300 and subsequently less than 100. Hence we may confine our attention to the 54 digit pairs formed from digits 0, 1, $\cdots$, 9.

With these, it takes but a few minutes to show that successive operations do lead to 1 or 4 for some n.

Also solved by L. Carlitz, Duke University; Daniel I. A. Cohen, *Philadelphia, Pennsylvania; David L. Silverman, Beverly Hills, California; W. C. Waterhouse, Harvard University; and the proposer.*

Comments on Solutions

446. [May 1961 and January 1962]

J. A. H. Hunter, Toronto, Canada, pointed out that the word "alphametic" is a noun. It is used to designate those cryptarithms that are made up of meaningful words. The "doubly true" cryptarithm in Problem 446 is indeed an alphametic. Examples of other alphametics which have appeared elsewhere include:

COOK	*APE*	*Q UAIL*
TOOK	*UP*	*P RAY*
BOOK	*XXXX*	*E AT*
TOKEN	*XAX*	*UP*
	ROPE	

451. [May 1961 and January 1962]

B. L. Schwartz, the proposer, made the following comments and alternate solution.

The solution given meets only requirement (b). There would be little point in submitting a problem which had only that as its objective, for the result is elementary and is established in many texts. See, for example, Olmstead, *Advanced Calculus* pp. 404-5, where essentially the same proof as Brown's is given. But the point of the proposal is the conditional convergence of the subseries. As a counterexample to the published solution, consider the series whose terms are

$$x_{2n-1} = \frac{(-1)^{n+1}}{2^n} \qquad \text{for } n = 1, 2, 3, \cdots ,$$

$$x_{2n} = \frac{(-1)^{n+1}}{n} \qquad \text{for } n = 1, 2, 3, \cdots ,$$

when $r = 1/3$.

A solution which satisfies both requirements (a) and (b) follows.

Let $x_n^+ = \max(x_n, 0)$, $x_n^- = \min(x_n, 0)$. Without loss of generality, we assume $r > 0$. Choose minimum k_1 such that

$$S_1 = \sum_0^{k_1} x_n^+ > r + 1 .$$

Now choose minimum k_2 such that

$$S_2 = S_1 + \sum_{k_1+1}^{k_2} x_n^- < r - 1 .$$

Next choose minimum k_3 such that

$$S_3 = S_2 + \sum_{k_2+1}^{k_3} x_n^+ > r + \frac{1}{2};$$

again choose minimum k_4 such that

$$S_4 = S_3 + \sum_{k_3+1}^{k_4} x_n^- < r - \frac{1}{2}.$$

Continue in this manner inductively. Choose k_{2i+1} as the minimum such that

$$S_{2i+1} = S_{2i} + \sum_{k_{2i}+1}^{k_{2i+1}} x_n^+ > r + \frac{1}{i+1}.$$

Then choose k_{2i+2} as the minimum such that

$$S_{2i+2} = S_{2i+1} + \sum_{k_{2i+1}+1}^{k_{2i+2}} x_n^- < r - \frac{1}{i+1}.$$

Consider

$$S' = \sum_{1}^{k_1} x_n^+ + \sum_{k_1+1}^{k_2} x_n^- + \sum_{k_2+1}^{k_3} x_n^+ + \sum_{k_3+1}^{k_4} x_n^- + \cdots.$$

Since both x_{k_i} and $\frac{1}{i+1}$ have limits zero, it is clear that S' has limit r. Furthermore,

$$\sum_{k_{2i}+1}^{k_{2i+1}} x_n^+ > \frac{1}{i+1}$$

since $S_{2i} < r$. Hence, the subseries comprising only the positive terms dominates $\Sigma \frac{1}{i+1}$. A similar argument shows that the subseries consisting of all the negative terms also must diverge. The result follows.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 297. A circle is divided into n equal parts. Every division point is then connected to every division point m steps away (but these lines are not diameters). Prove that not more than two of these lines pass through any interior point of the circle. [*Submitted by Brother U. Alfred.*]

Q 298. Which is greater $(n!)^2$ or n^n? [*Submitted by C. W. Trigg.*]

Comment on a Quickie

Q 232. [November 1958, September 1959] Find an integral root of

$$(x-1)(x-3)(x-4)(x-5) = 360 .$$

Alternate solution by C. W. Trigg. Any rational root will convert the left-hand member into the product of four consecutive integers.

$$360 = 6 \cdot 5 \cdot 4 \cdot 3 = (-3)(-4)(-5)(-6) ,$$

so $x = 8$ and $x = -1$ are two roots. The sum of the other two roots is $(2+3+4+5-8+1)$ or 7 and their product is $(2 \cdot 3 \cdot 4 \cdot 5 - 360)/(8)(-1)$ or 30. Hence the other two roots are

$$\frac{7 \pm \sqrt{7^2 - 4(30)}}{2} \quad \text{or} \quad \frac{7 \pm \sqrt{-71}}{2} .$$

This may be generalized to

$$[x-a][x-(a+1)][x-(a+2)][x-(a+3)] = b(b+1)(b+2)(b+3) .$$

Then $x = a+b+3$ and $a-b$. The sum of the other two roots is $(2a+3)$ and their product is $a^2+b^2+3(a+b)+2$. Hence the other two roots are

$$(2a+3) \pm \frac{\sqrt{1-4b(b+3)}}{2} ,$$

and are complex except in the trivial case where the right-hand member is zero.

(Answers to Quickies are on page 172.)

last three moments to 3, 7, 17, and start once more, getting the new moments: 36, 80, 176, 384, 832 to get the approximation 2.17.

In summary, to approximate the zero of a polynomial which is farthest from (nearest to) h, expand the reciprocal of that polynomial in negative (positive) powers of $x-h$ and use the quotient of the coefficients of two consecutive powers of $x-h$, the coefficient of the greater power being in the denominator, as an approximation. Synthetic division may be used to determine the coefficients. This is the viewpoint which is probably most adaptable to an elementary algebra course. Note that one could then present this as a possible method of developing tables of roots, exponential tables ($10^{.3}$ is a solution of $x^{10}-1000=0$, and one might find the zero nearest 2) and trigonometric tables (with the aid of De Moivre's theorem).

Finally, the alternate viewpoint has been that of using a particular solution of a linear homogeneous difference equation with constant coefficients to approximate the zeros of its auxiliary polynomial. This is the reverse procedure to that usually found in books on finite differences and numerical analysis. These approximate solutions to the zeros of the polynomial could now be used to get approximations to the general solution of the difference equation. Both methods have generated moments, a procedure used in statistics.

Note: A. Ostrowski deals with this method of approximating zeros of a polynomial from the complex variables viewpoint in Appendix J of *Solutions of Equations and Systems of Equations,* Academic Press, 1960.

Answers *to Quickies on page 193.*

A 297. The lines are equal chords tangent to a circle concentric with the given circle. If more than two lines passed through one point, there would be more than two tangents from an external point to a circle, which is impossible.

A 298. We have $(1!)^2 = 1^1$ and $(2!)^2 = 2^2$, but for $n > 2$ we have

$$\frac{(n!)^2}{n^n} = \frac{n\cdot 1}{n}\cdot\frac{(n-1)2}{n}\cdot\frac{(n-2)3}{n}\cdots\frac{2(n-1)}{n}\cdot\frac{1\cdot n}{n} > 1$$

since all of the fractions in the product are $\geqq 1$, and at least one fraction exceeds 1. Therefore $(n!)^2 > n^n$ for $n > 2$.

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